Tutorial XIII

14 May 2020 14:02

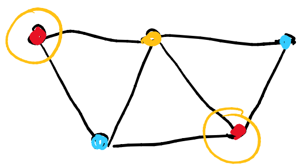
THE PROBABILISTIC METHOD II

1.) Design a randomized experiment in which the desired object is created

2.) if $\Pr(\text{object with desired properties}) > 0 \Rightarrow$ existence of desired object.

Show existence of graphs with large girth (ℓ) and chromatic number (χ)

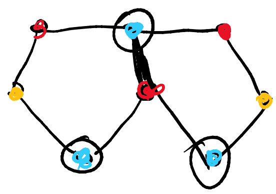
Graph $G=(V,E)$ has girth ℓ, if there are no cycles smaller than ℓ.



$\ell = 3$

$\chi = 3$

$\alpha = 2$



$\ell = 5$

$\chi = 3$

$\alpha = 3$

G: $a, b \in \mathbb{N}$ $\quad \ell(G) > a$ $\quad \chi(G) > b$

??

Chromatic number (χ) - the smallest number of colors for vertices, such that no edge connects two vertices of the same color.

Independence number (α) of graph G is the size of the largest independent set of vertices — without any edges between them.

$$\alpha(G) \geq \frac{|V|}{\chi(G)} \qquad \chi(G) \geq \frac{|V|}{\alpha(G)}$$

Intuition on why finding graphs with large χ and large ℓ is difficult

→ In order to avoid small cycles the number of edges is rather small

→ Small number of edges leads to large inde

→ Small number of edges leads to large independence number

→ large independence number implies small chromatic number

Approach → create a random graph (n-vertices and each of $\binom{n}{2}$ edges is added with probability p)

We will show that for sufficiently large n and suitably chosen p, graph G with $\ell(G) > a$ and $\chi(G) > b$ exists (gets constructed w.p. larger than 0).

We will split this into two events:

1.) the probability that the number of small cycles ($< \ell$) is large is smaller than $1/2$

$\boxed{E_1 - \text{number of small cycles is large}}$

2.) the probability of a large independence number is smaller than $1/2$

$\boxed{E_2 - \text{independence number is large}}$

We want a graph with neither of those properties:

$\Pr(E_1) < 1/2$ $\Pr(E_2) < 1/2$

$$\Pr(\neg E_1 \wedge \neg E_2) = 1 - \Pr(E_1 \vee E_2) \geq 1 - (\Pr(E_1) + \Pr(E_2)) > 0$$

due to Union bound

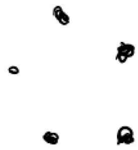
Random graph - add each edge with probability

$p = n^{\lambda - 1}$ $\quad \lambda \in (0, 1/\ell)$ $\quad$ [importantly $\lambda \cdot \ell < 1$]

We want the probability that the number of cycles of size $\leq \ell$ is larger than $\frac{n}{2}$ to be smaller than $1/2$.

Let X be the number of cycles smaller than ℓ

$$\Pr\left(X > \frac{n}{2}\right) = ?$$



$\binom{5}{3}$ possible cycles but probabilities to get them are not independent

In order to get around the dependence of the cycles, we will evaluate $E(X)$ instead and use Markov's inequality

$$P(X > t) \leq \frac{E(X)}{t}$$

if we show $\boxed{E(X) < \frac{n}{4}}$ then by M.I. $\Pr(X > \frac{n}{2}) < 1/2$

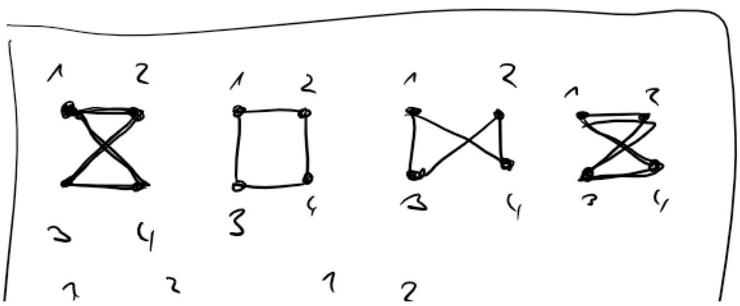
In order to calculate $E(X)$ define $N_{x_1, \dots, x_j} = 1$ when vertices $x_1, \dots, x_j$ form a cycle

$= 0$ otherwise

$$X = \sum_{j=3}^{\ell} \sum_{j\text{-tuples}} N_{x_1, \dots, x_j}$$

$$E(X) = \sum_{j=3}^{\ell} \sum_{j\text{-tuples}} \Pr(N_{x_1, \dots, x_j} = 1)$$

p

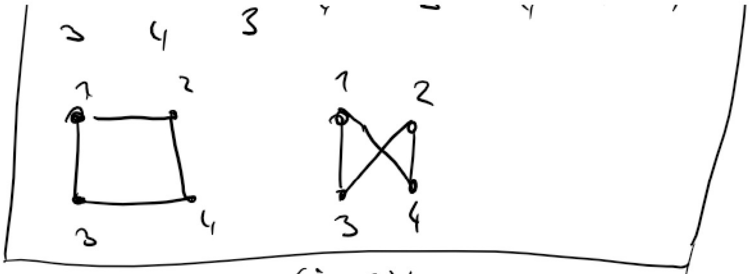


$$\Pr(N_{x_1, \dots, x_j} = 1) = p^j$$

ℓ [redacted] .

$i_1, \dots, i_j$

$$E(X) = \sum_{j=3}^{\ell} \boxed{\binom{n}{j} \frac{(j-1)!}{2}} \cdot p^j$$



j-tuple has $\frac{(j-1)!}{2}$ unique cycles

$$\frac{n!}{(n-j)!\, \cancel{j!}} \cdot \frac{\cancel{(j-1)!}}{2} = \frac{n!}{(n-j)!\, 2j}$$

$$= \frac{n \cdot (n-1) \cdot \ldots \cdot (n-j+1)}{2j} < n^j$$

(for sufficiently large n)

$$< \sum_{j=3}^{\ell} n^j p^j$$

$$= \sum_{j=3}^{\ell} n^j \left(n^{\lambda-1}\right)^j$$

$$= \sum_{j=3}^{\ell} n^j \cdot n^{\lambda j} \cdot n^{-j}$$

$$= \sum_{j=3}^{\ell} n^{\lambda j}$$ $\leftarrow$ geometric series with gradient n^λ

$$\boxed{\sum_{i=0}^{n} a^i = \frac{1-a^{i+1}}{1-a}}$$

$$< \sum_{j=0}^{\ell} n^{\lambda j}$$

$$= \frac{1-(n^\lambda)^{\ell+1}}{1-(n^\lambda)} = \frac{(n^\lambda)^{\ell+1}-1}{(n^\lambda)-1} < \frac{n^{\lambda\ell} \cdot n^\lambda}{n^\lambda - 1} = \frac{n^{\lambda \ell}}{1-n^{-\lambda}} < \frac{n}{4}$$

for sufficiently large n

lets show that there is n_c, s.t. $\forall n > \boxed{n_c}$

$$\frac{n^{\lambda\ell}}{1-n^{-\lambda}} < \frac{n}{c}$$ (c is an arbitrary positive number)

$$n^{\lambda\ell} < \frac{n}{c} \cdot \left(1-n^{-\lambda}\right)$$

$$< \frac{n}{n} - \frac{n^{(1-\lambda)}}{c}$$

$$\boxed{n^{\lambda\ell}} + \frac{\boxed{n^{(1-\lambda)}}}{} < \frac{\boxed{n}}{}$$

$$n^{\lambda \ell} + \frac{n^{(1-\lambda)}}{c} < \frac{n}{c}$$

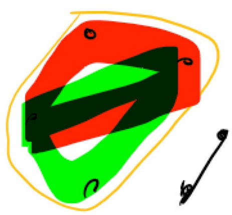
r.h.s. increases more rapidly assymptotically than l.h.s.

$E(X) < \frac{n}{4}$ for sufficiently large n

$\Rightarrow \Pr\left(X > \frac{n}{2}\right) < \frac{1}{2}$

2.) Independence number $\alpha(G)$ is small

m = 3



$$\Pr(\alpha(G) \geq \underset{\text{specified later}}{m}) < \frac{1}{2}$$

$$\leq \sum_{S \subset V, |S| = m} \Pr(S \text{ is an independent set})$$

$$= \binom{n}{m} (1-p)^{\binom{m}{2}}$$

$\binom{n}{m} \leq n^m$

$(1-x) < e^{-x} \Rightarrow (1-p) < e^{-p}$

$$< n^m e^{-p \frac{m(m-1)}{2}}$$

$m = \left\lceil \frac{3}{p} \cdot \ln(n) \right\rceil$

$e^{\ln n} = n$

$$< n^m n^{-p \frac{3}{p} \frac{m-1}{2}}$$

$$= n^m n^{-\frac{3(m-1)}{2}}$$

$$= n^{\frac{2m - 3m + 3}{2}}$$

$$3 - m \quad - \quad \frac{3}{2} - \frac{3}{2p} \cdot \ln(n)$$

$$= n^{\frac{3-m}{2}} \leq n^{\frac{3}{2} - \frac{3}{2p} \cdot \ln(n)}$$

$$= n^{\frac{3}{2} - \frac{3}{2} \cdot \frac{\ln(n)}{n^{(k-1)}}}$$

$$\simeq n^{-\frac{\ln(n)}{n^{-c}}} \qquad c \in \left(-1, -\frac{l-1}{l}\right)$$

$$\overset{n \to \infty}{\sim} 0$$

Probability that $\alpha(G) > \lceil \frac{3}{p} \ln(n) \rceil$ is smaller than $\frac{1}{2}$ for sufficiently large n

To sum up:

The probability to construct a graph G with the number of cycles of size $\leq l$ smaller than $\frac{n}{2}$ and independance number smaller than $\lceil \frac{3}{p} \ln(n) \rceil$ is positive $\Rightarrow$ IT EXISTS!

From G you can construct G' by deleting a vertex from each cycle.

G' - no cycles $\leq l$

$$\chi(G') > \frac{|V(G')|}{\alpha(G')} > \frac{n/2}{\frac{3 \cdot n^{1-\lambda} \cdot \ln(n)}{2}} \overset{n \to \infty}{\rightsquigarrow} \infty$$

There are graphs with arbitrary girth l and chromatic number χ.