Markov Models

PA154 Jazykové modelování (5.1)

Pavel Rychlý

pary@fi.muni.cz

March 16, 2020

Source: Introduction to Natural Language Processing (600.465)
Jan Hajič, CS Dept., Johns Hopkins Univ.
www.cs.jhu.edu/˜hajic

Review: Markov Process

- Bayes formula (chain rule):

$$P(W) = P(w_1, w_2, .., w_T) = \Pi_{i=1..T}\, p(w_i \mid \cancel{w_1, w_2, ..}, w_{i-n+1}, .., w_{i-1})$$

- n-gram language models:
  - Markov process (chain) of the order n-1:

approximation

$$P(W) = P(w_1, w_2, .., w_T) = \Pi_{i=1..T}\, p(w_i \mid w_{i-n+1}, w_{i-n+2}, .., w_{i-1})$$

Using just <u>one</u> distribution (Ex.: trigram model: $p(w_i \mid w_{i-2}, w_{i-1})$) :

Positions:	1	2	<u>3</u>	<u>4</u>	<u>5</u>	6	7	8	9	10	11	<u>12</u>	<u>13</u>	<u>14</u>	15	16
Words:	My	car	*broke*	*down*	*,*	and	within	hours	Bob	's	can	*broke*	*down*	*,*	too	.

$$\text{p}(, \mid \text{broke down}) = p(w_5 \mid w_3, w_4) = p(w_{14} \mid w_{12}, w_{13})$$

Markov Properties

- Generalize to any process (not just words/LM):
  - Sequence of random variables: $X = (X_1, X_2, \ldots, X_T)$
  - Sample space S (*states*), size N: $S = (S_0, S_1, S_2, \ldots, S_N)$

1. Limited History (Context, Horizon):

   $\forall i \in 1..T; P(X_i \mid X_1, .., X_{i-1}) = (X_i \mid X_{i-1})$

   1 7 3 7 9 0 6 $\boxed{3}$ 4 5. . .   1 7 3 7 9 0 6 7 $\boxed{3}$ 4 5. . .

2. Time invariance (M.C. is stationary, homogenous)

   $\forall i \in 1..T, \forall y, x \in S; P(X_i = y \mid X_{i-1} = x) = p(y \mid x)$

   1 $\boxed{7}$ $\boxed{3}$ $\boxed{7}$ $\boxed{9}$ 0 6 $\boxed{7}$ $\boxed{3}$ 4 5 . . .

   ?   ok. . . same **distribution**

Long History Possible

- What if we want trigrams:

  $1\ 7\ 7\ 3\ 7\ 9\ 0\ \boxed{6\ 7}\ \boxed{3}\ 4\ 5\ldots$

- Formally, use transformation:

  Define new variables $Q_i, such that X_i = Q_{i-1}, Q_i$:

  Then

$$P(X_i \mid X_{i-1}) = P(Q_{i-1}, Q_i \mid Q_{i-2}, Q_{i-1})$$

Predicting (X_i) $1\ 7\ 3\ 7\ \boxed{9\ 0}\ 6\ 7\ 3\ 4\ 5\ldots$

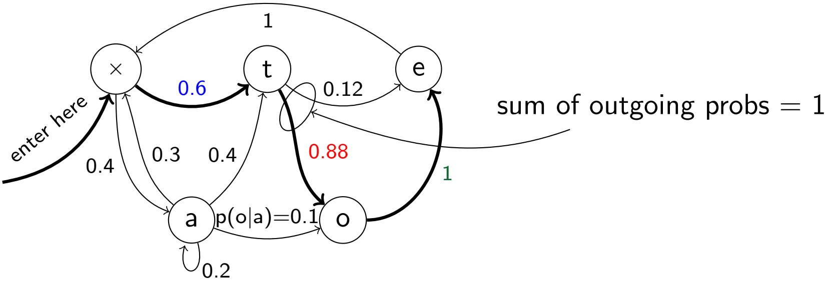
History $(X_i = \{Q_{i-2}, Q_{i-1}\})$:

x	1	7	3	...	0	6	7	3	4
x	x	1	7	...	9	0	6	7	3

Graph Representation: State Diagram

- $S = \{s_0, s_1, s_2, \ldots, s_n\}$: states
- Distribution $P(X_i \mid X_{i-1})$:
  - transitions (as arcs) with probabilities attached to them:

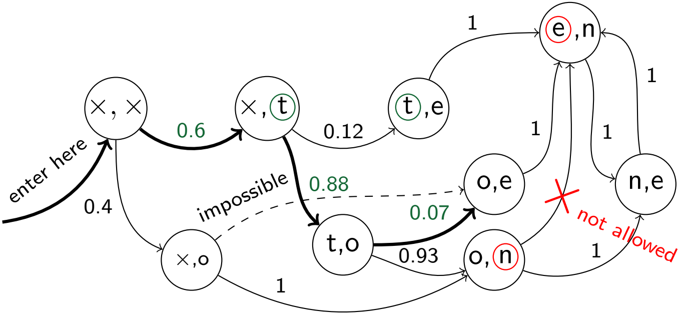
Bigram case:



$p(\text{toe}) = .6 \times .88 \times 1 = .528$

The Trigram Case

- S = $\{s_0, s_1, s_2, \ldots, s_n\}$: states: pairs s_i = (x,y)
- Distribution $P(X_i \mid X_{i-1})$: (r.v. X: generates pairs s_i)



p(toe) = .6 × .88 × .07 ≅ .037

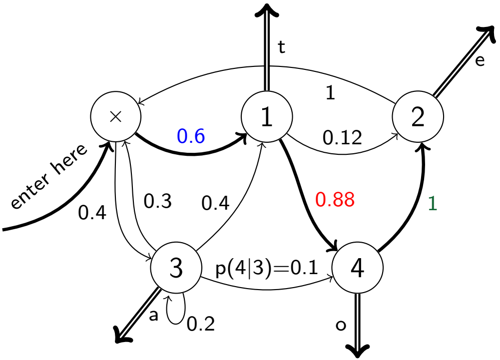
p(one) = ?

Finite State Automaton

- States $\sim$ symbols of the [input/output] alphabet
  - pairs (or more): last element of the n-tuple
- Arcs $\sim$ transitions (sequence of states)
- [Classical FSA: alphabet symbols on arcs:
  - transformation: arcs $\leftrightarrow$ nodes]
- Possible thanks to the "limited history" Markov Property
- So far: **Visible** Markov Models (VMM)

Hidden Markov Models

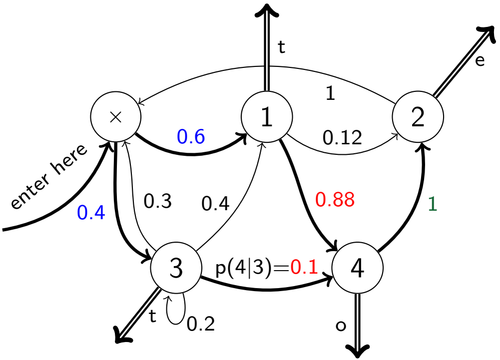
- The simplest HMM: states generate [observable] output (using the "data" alphabet) but remain "invisible":



$$p(\text{toe}) = .6 \times .88 \times 1 = .528$$

Added Flexibility...

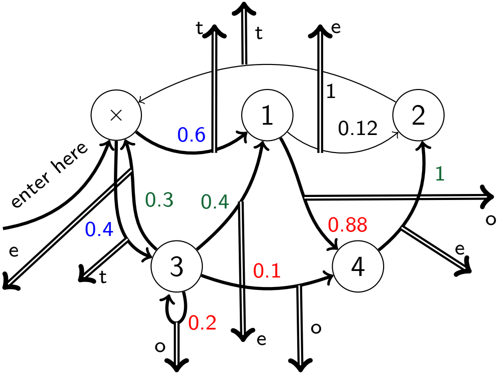
- So far, no change; but different states may generate the same output (why not?):



$$p(toe) = .6 \times .88 \times 1 + .4 \times .1 \times 1 = .568$$

Output from Arcs...

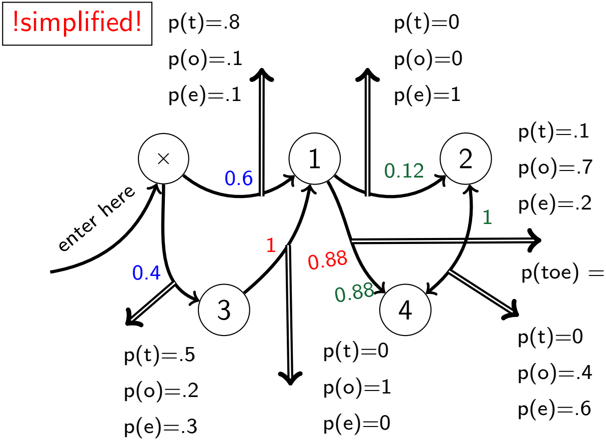
- Added flexibility: Generate output from arcs, not states:



$$
\begin{aligned}
p(\text{toe}) = {} & .6 \times .88 \times 1 + \\
& .4 \times .1 \times 1 + \\
& .4 \times .2 \times .3 + \\
& .4 \times .2 \times .4 = .624
\end{aligned}
$$

...and Finally, Add Output Probabilities

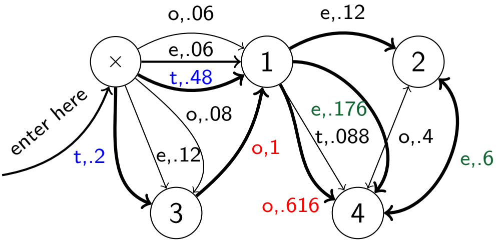
- Maximum flexibility: [Unigram] distribution (sample space: output alphabet) at each output arc:



$$\begin{aligned} p(toe) = {} & .6 \times .8 \times .88 \times .7 \times 1 \times .6 \; + \\ & .4 \times .5 \times 1 \times 1 \times .88 \times .2 \; + \\ & .4 \times .5 \times 1 \times 1 \times .12 \times 1 \\ & \cong .237 \end{aligned}$$

Slightly Different View

- Allow for multiple arcs from $s_i \rightarrow s_j$, mark them by output symbol s, get rid of output distributions:



$$\begin{aligned} p(\text{toe}) = {} & .48 \times .616 \times .6 + \\ & .2 \times 1 \times .176 + \\ & .2 \times 1 \times .12 \cong .237 \end{aligned}$$

In the future, we will use the view more convenient for the problem at hand.

Formalization

HMM (the most general case):

- five-tuple (S, s_0, Y, P_S, P_Y), where:
  - S $= \{s_0, s_1, s_2, \ldots, s_T\}$ is the set of states, s_0 is the initial state,
  - Y $= \{y_1, y_2, \ldots, y_V\}$ is the output alphabet,
  - $P_S(s_j \mid s_i)$ is the set of prob. distributions of transitions,
    - size of $P_s :\mid S \mid^2$.
  - $P_Y(y_k \mid s_i, s_j)$ is the set of output (emission) probability distributions.
    - size of $P_Y :\mid S \mid^2 x \mid Y \mid$

Example:

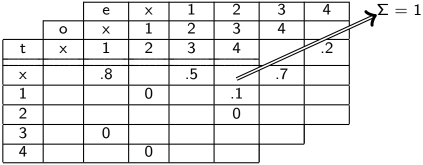
- S$=$ x, 1, 2, 3, 4, $s_0 =$ x
- Y $= \{t, o, e\}$

- Example (for graph, see foils 11,12):
  - S= $\{x, 1, 2, 3, 4\}$, $s_0 = x$
  - Y= $\{e, o, t\}$
  - P_S :

	x	1	2	3	4
x	0	.6	0	.4	0
1	0	0	.12	0	.88
2	0	0	0	0	1
3	0	1	0	0	0
4	0	0	1	0	0

$\longrightarrow \Sigma = 1$

P_Y:



Using the HMM

- The generation algorithm (of limited value :-)):
  1. Start in $s = s_0$.
  2. Move from s to s' with probability $P_S(s' \mid s)$.
  3. Output (emit) symbol yk with probability $P_S(y_k \mid s, s')$.
  4. Repeat from step 2 (until somebody says enough).
- More interestirig usage:
  - Given an output sequence $Y = \{y_1, y_2, \ldots, y_k\}$ compute its probability.
  - Given an output sequence $Y = \{y_1, y_2, \ldots, y_k\}$ compute the most likely sequence of states which has generated it.
  - ... plus variations: e.g., n best state sequences