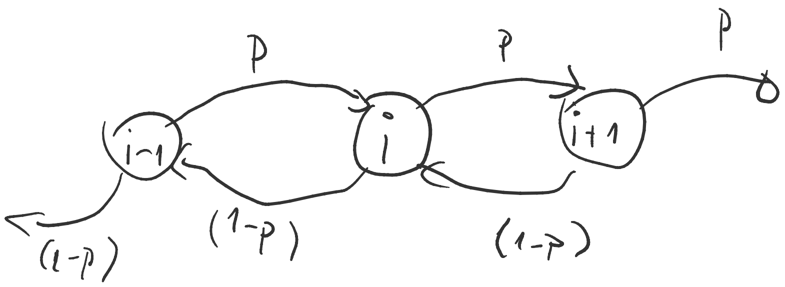
Tutorial 11

13 May 2021 16:01

MARKOV CHAINS II

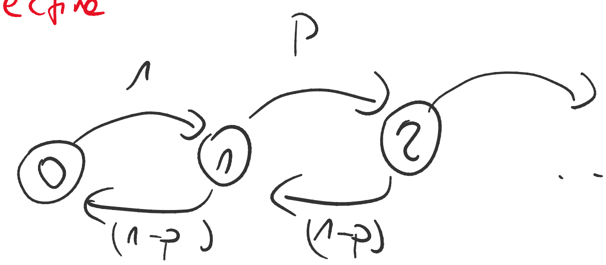
- Walks on a line
- Randomized algorithm for 2-SAT (3-SAT)
- Fair 2-colorability of 3-colorable graphs

Walks on a line

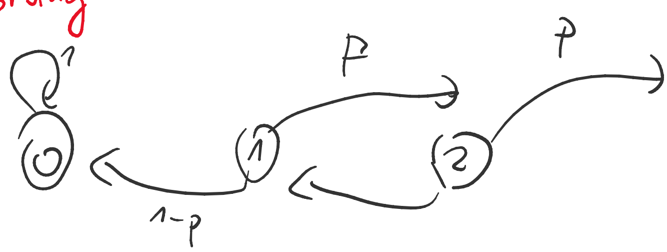


If the line is not infinite in both directions it contains at least one barrier.

Reflective

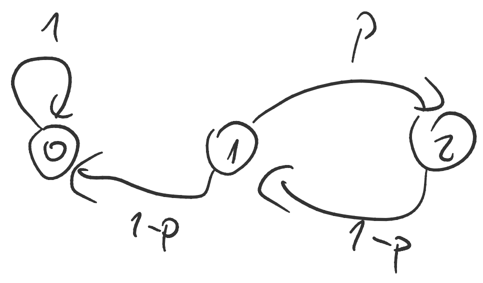


Absorbing

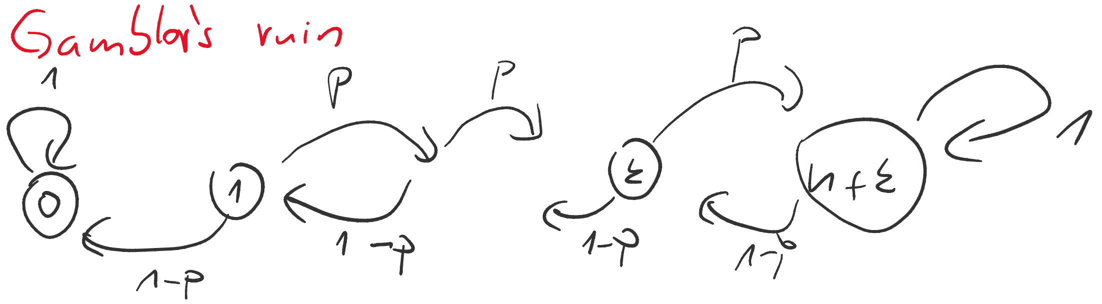


Man on a cliff

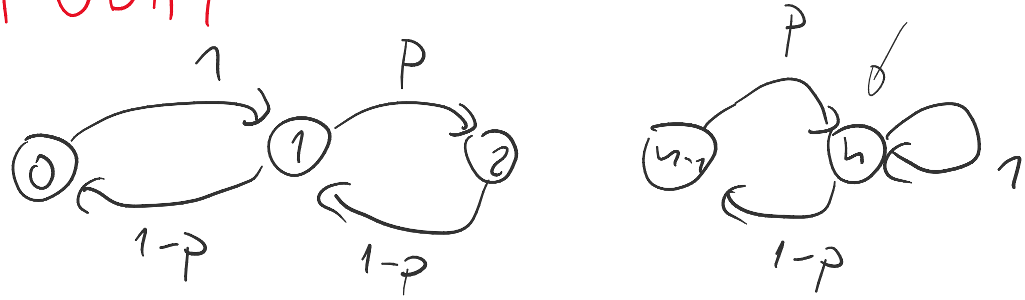
Monkey on a cliff



Gambler's ruin



TODAY



→ What is the expected time to get from 0 to n.

$E_{i,j}$ = expected time to reach j from i

$E_{j,j} = 0$

$\boxed{E_{01} = 1}$

$$\boxed{E_{0,n} = E_{0,1} + E_{1,2} + E_{2,3} + \dots + E_{n-1,n}}$$

$\forall i \quad E_{i,i+2} = E_{i,i+1} + E_{i+1,i+2}$

$$\forall i \quad E_{i,i+2} = E_{i,i+1} + E_{i+1,i+2}$$

$$\forall i \; E_{i,i+1} = 1 + p \cdot \overbrace{E_{i+1,i+1}}^{=0} + (1-p)\, E_{i-1,i+1}$$

$$= 1 + (1-p)\left(E_{i-1,i} + E_{i,i+1}\right)$$

$$= 1 + (1-p)E_{i-1,i} + (1-p)\,E_{i,i+1}$$

$$p\,E_{i,i+1} = 1 + (1-p)\,E_{i-1,i}$$

$$E_{i,i+1} = \frac{1}{p} + \frac{(1-p)}{p} E_{i-1,i}$$

$$E_{i,i+1} = r_i$$

$$\boxed{\begin{aligned} r_0 &= 1 \\ r_i &= 1/p + \frac{(1-p)}{p} \underline{r_{i-1}} \end{aligned}}$$

This is called a linear recursive relation.

For given p solutions are easy to find.

Example for $p = 1/2$

$$\boxed{\begin{aligned} r_0 &= 1 \\ r_i &= 2 + \cdot r_{i-1} \end{aligned}} \Rightarrow \underline{r_i = 2i + 1}$$

$$\boxed{\begin{aligned} r_0 &= c_1 \\ r_1 &= c_2 \\ r_2 &= c_3 \\ r_{i+3} &= a_0 r_i + a_1 r_{i+1} + a_2 r_{i+2} \end{aligned}}$$

2-SAT

Logical formula with $x_1, \dots, x_n$ terms

$$\overline{(x_1 \vee x_2)} \wedge (x_3 \vee \neg x_2) \wedge \dots \wedge \overline{(\neg x_1 \vee \neg x_5)}$$

Is it satisfiable? → is there an assignment of truth values (0/1) to variables x_i such that all terms are satisfied?

1.) Assign values to variables at random

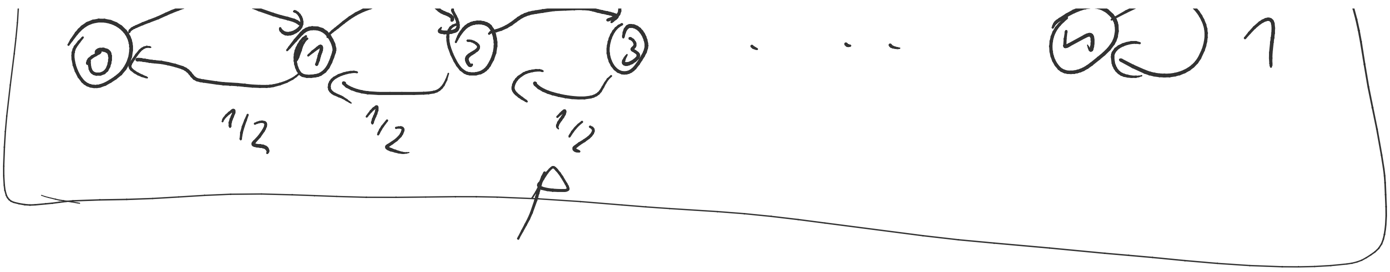
Repeat the following:

2.) find an unsatisfied term (if it doesn't exist, you have a solution) with variables x_a and x_b. Randomly choose x_a or x_b and flip its assignment.

This is a Las Vegas algorithm. If it runs too long output 'IDK'

if A (the assignment) exists how long will it take to find it?

We count the number of variables in an intermediate solution (before step 2) with assignment identical to A.



$(x_a \vee x_b) \rightarrow$ if this is not satisfied AT LEAST one of the two variables has a value different to A.

$\rightarrow A$ 0 1

Therefore w.p. $\boxed{\text{AT LEAST}}$ $1/2$ random flipping gets us closer to A.

The $\boxed{\text{upper bound}}$ on the expected number of steps is equal to $E_{0,n}$.

$$\boxed{\begin{aligned} &E_{0,1} = 1 \\ &E_{i,i+1} = \frac{1}{p} + \frac{(1-p)}{p} E_{i-1,i} \\ &p = 1/2 \\ &E_{0,1} = 1 \\ &E_{i,i+1} = 2 + E_{i-1,i} \\ &E_{i,i+1} = 2i + 1 \end{aligned}}$$

$$E_{0,n} = \sum_{i=0}^{n-1} E_{i,i+1} = \sum_{i=0}^{n-1} 2i+1$$

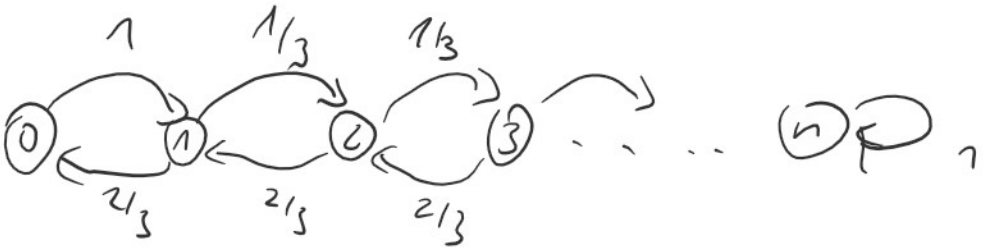
$$= n + 2 \cdot \sum_{i=0}^{n-1} i = n + 2 \cdot (n-1)\frac{(n-2)}{2} \in O(n^2)$$

The complete algorithm

Run the procedure $2n^2$ times. If the solution is not found say 'I don't know'

Why doesn't this work for 3-SAT?

$(x_a \vee x_b \vee x_c) \rightarrow$ terms now have 3 variables



AT LEAST 1 variable has a wrong assignment

therefore probability to improve is AT LEAST 1/3.

$$E_{0,1} = 1$$

$$E_{i,i+1} = \frac{1}{p} + \frac{1-p}{p} E_{i-1,i}$$

for $p = 1/3$

$$r_0 = 1$$

$$r_i = 3 + 2r_{i-1}$$

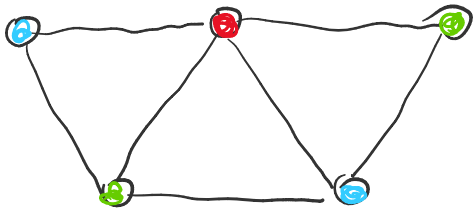
Solution $\boxed{r_i = 2^{i+2} - 3}$

$\Downarrow$

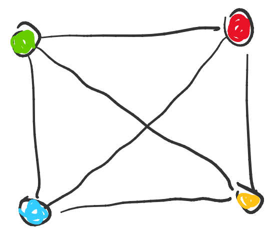
Expected number of steps to find a solution is exponential.

Let G be 3-colorable graph

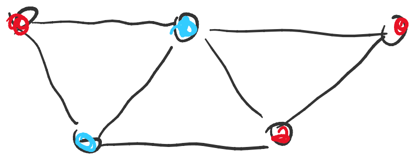
3 colorable example



G not 3-colorable



Task: Find a 2-coloring of G, such that there is no monochromatic triangle (fair 2-coloring).



It exists: 3-coloring to fair 2-coloring ? → Choose one of the colors and change it to one of the remaining 2 colors

→ $\boxed{\text{Choose one}}$ of the three colors.

To each node with this color assign one of the two remaining colors at random.

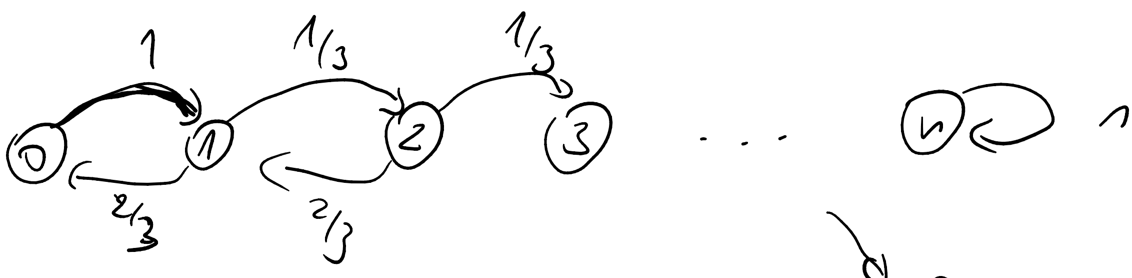
Randomized procedure

1.) Choose a random 2-coloring

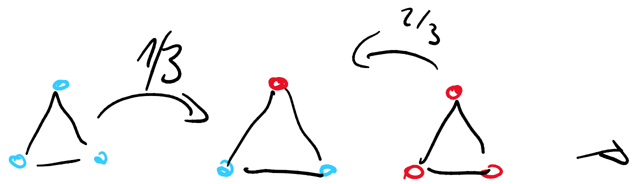
→ (if there are no such triangle

2.) Find a monochromatic triangle (if there are no such triangle we have a solution)

flip color of one of the nodes chosen at random.



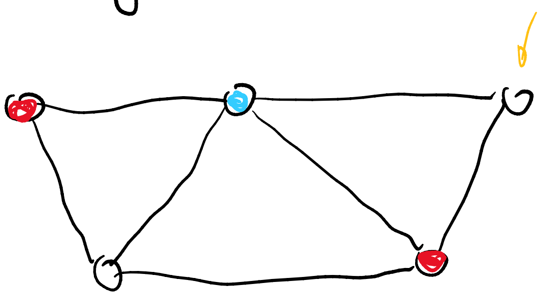
Same as 3-SAT!



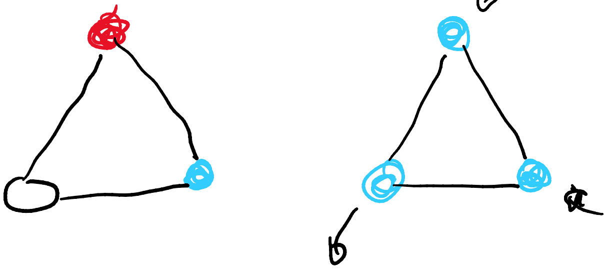
$P_{\text{worst case}}$ coloring A

What is important is not the correct 2-coloring itself, but the <u>partial 2-coloring</u> one obtains from the 3-coloring

I don't care about this color

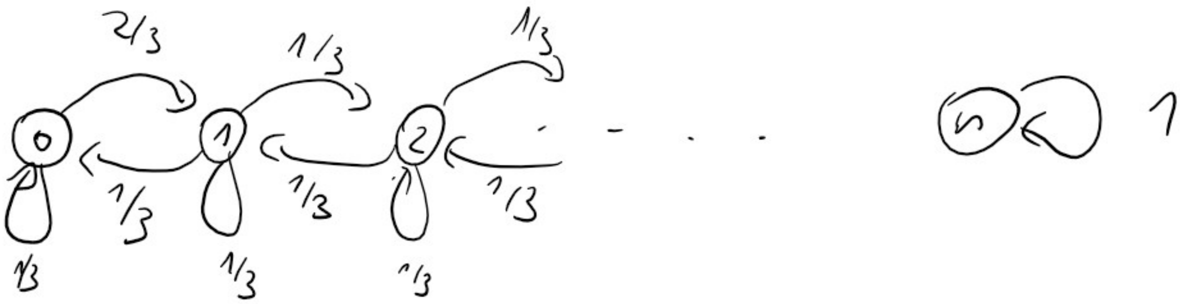


I get closer



I get further

this does not
change the distance!



$$E_{0,1} = 3/2$$

$$E_{i,i+1} = 3 + E_{i-1,i}$$

$$r_0 = 3/2 \qquad r_i = 3i + 3/2$$

$$r_i = 3 + r_{i-1}$$

$$E_{0n} = \sum_{i=0}^{n-1} E_{i,i+1} = \frac{3n}{2} + 3\sum_{i=0}^{n-1} i \in O(n^2)$$