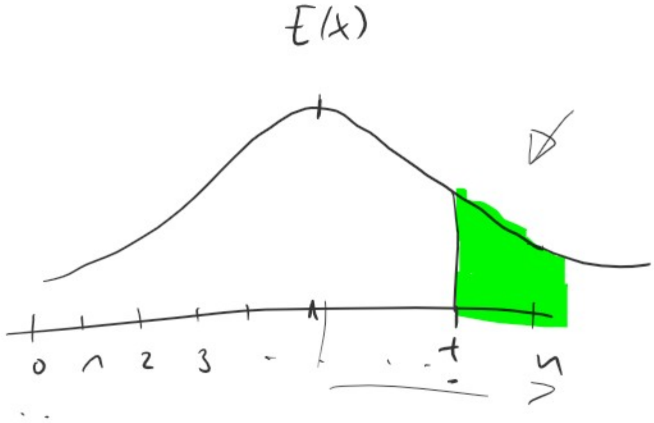
TAIL INEQUALITIES

18 March 2021 15:49

MARKOV'S INEQUALITY
CHEBYSHEV'S INEQUALITY
CHERNOFF'S INEQUALITY

MARKOV'S INEQUALITY



X - random variable with positive values.

→ E(X) - expected value

$$\Pr(X \geq t) \leq \frac{E(X)}{t} \leq 1 + \dots$$ if $t \leq E(X)$

$\boxed{t > E(X)}$

$\boxed{\text{EXAMPLE}}$ LV1 - always gives a correct answer in expected polynomial running time (in n - size of the input)

LV2 - always runs in polynomial time (in n - size of the input). But sometimes gives result "I don't know". (with fixed probability $p < 1$, typically $\boxed{(1-p) \geq 1/2}$.

Run LV1 for some time $[2E(n)+1]$ if it doesn't finish, terminate the run and answer "I don't know".

The probability p of "I don't know" answer can be calculated as follows:

X_n - number of steps needed with LV1 on input size (n)

$E(X_n)$ $E(X_n)$

$$\Rightarrow \Pr\big(X_n \geq \underbrace{2E(X_n)+1}_{t}\big) \leq \frac{E(X_n)}{t} \leq \frac{E(X_n)}{2E(X_n)+1} < 1/2$$

Probability of "I don't know" answer.

Chebyshev's inequality

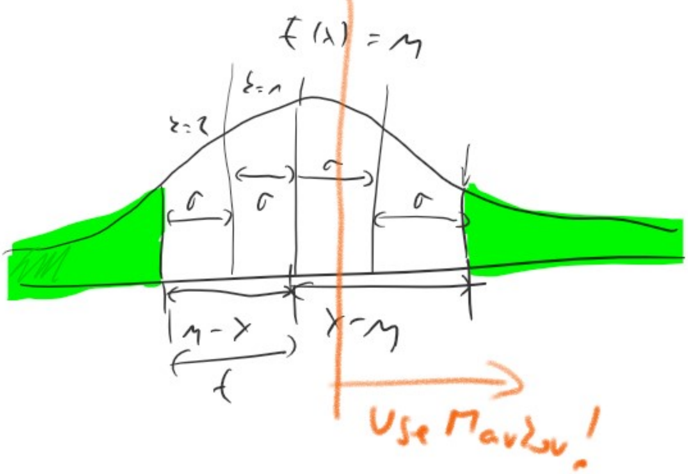
X has a finite $E(X) = \mu$

$Var(X) = \sigma^2$ σ - standard deviation

$$\Pr(|X-\mu| \geq \underbrace{k\sigma}_{t}) \leq \frac{1}{k^2} \leftarrow$$

$$\Updownarrow$$

$$\Pr(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}$$



Useful only for $k > 1$

Chernoff's bound

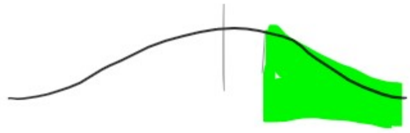
Specific form of random variables i.i.d.

$X = \sum_{i=1}^{n} X_i$, where X_i are identicaly and independenty distributed binary random variables with $\Pr(X_i = 1) = p$.

$$\mu = E(X) = n \cdot p$$

e - Euler's number

$$\Pr(X > (1+\delta)\mu) < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$



$$\Pr(X > (1+\delta)\mu) < \left(\frac{\quad}{(1+\delta)^{(1+\delta)}}\right)$$

$$\Pr(X < (1-\delta)\mu) < \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu}$$

↑ Hard to use

Simpler (and looser) expressions are

$$\Pr(X \leq (1-\delta)\mu) \leq e^{-\frac{\delta^2\mu}{2}} \leq e^{-\frac{\delta^2\mu}{3}} \qquad 0 < \delta < 1$$

$$\Pr(X \geq (1+\delta)\mu) \leq e^{-\frac{\delta^2\mu}{3}} \qquad 0 < \delta < 1$$

$$\Pr(|X-\mu| \geq \delta\mu) \leq 2e^{-\frac{\delta^2\mu}{3}}$$

$$= \Pr(X-\mu \geq \delta\mu) \vee \Pr(\mu - X \geq \delta\mu)$$

$$X \geq \delta\mu + \mu$$

$$X \geq (1+\delta)\mu$$

EXAMPLE

In the experiment we roll a 6-sided die n times

X is the number of outcomes '6' in our experiment.

Calculate (or estimate) $\Pr(X \geq n/4) = \sum_{i \geq n/4}^{n} \binom{n}{i} \left(\frac{1}{6}\right)^{i} \left(\frac{5}{6}\right)^{n-i}$

↑ ↑

Markov's Inequality

$E(X) = n/6$

$$E(X) = n/6$$

$$Pr\left(X \ge \frac{n}{4}\right) \le \frac{E(X)}{n/4} = 2/3$$

Chebyshev's inequality

$$Var(X) = \sigma^2 = n \cdot p \cdot (1-p) = n \cdot 1/6 \cdot 5/6 = \frac{5n}{36}$$

$$Pr\left(X \ge \frac{n}{4}\right)$$

$$\|$$

$$Pr\left(|X - E(X)| \ge t\right) \le \frac{Var(X)}{t^2}$$

$$\|$$

$$Pr\left(X - E(X) \ge t\right) \vee Pr\left(E(X) - X \ge t\right) \le \frac{5n/36}{n^2/144} = \frac{20}{n}$$

$$\vee|$$

$$Pr\left(X - E(X) \ge t\right)$$

$$\|$$

$$Pr\left(X \ge E(X) + t\right)$$

$$n > 20$$

$$E(X) + t = n/4$$

$$t = n/4 - n/6$$

$$t = n/12$$

Chernoff bound

$$X = \sum_{i=1}^{n} X_i \qquad Pr(X_i = 1) = 1/6$$

$$Pr\left(X \ge (1+\delta)\mu\right) \le e^{-\frac{\delta^2 \mu}{3}}$$

$$(1+\delta)\mu = \frac{n}{4}$$

$$(1+\delta)\, y = \frac{n}{4}$$

$$(1+\delta)\, \frac{n}{6} = \frac{n}{4}$$

$$\delta = 1/2 \quad \checkmark$$

$$P_r\left(X \geq \left[(1+1/2)\frac{n}{6}\right]\right) \leq e^{-\frac{n}{72}}$$

Amplification of success probability for 2MC algorithms

BPP - Probability of a correct result $\geq 1/2 + \varepsilon$

PP - Probability of a correct result $> 1/2$ $\quad [1/2 + \delta(n)]$

Probability amplification = run the algorithm k-times and use majority voting $\quad \underbrace{(Y|N|N|Y|N|Y|Y|N|Y| \quad | \quad | \ldots |N)}_{\text{if } Y > N \text{ then } Y \text{ otherwise } N}$

Chernoff's bound

X_i - characterizes ith run

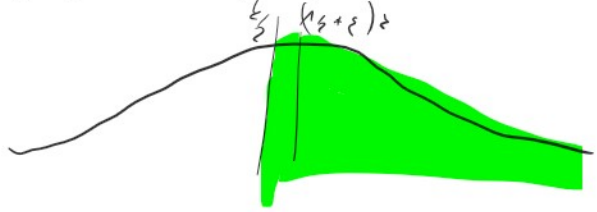
$X_i = 1$ if the correct answer is given

$X_i = 0$ if the incorrect answer is given

$X = \sum_{i=1}^{k} X_i$ $\quad$ -> number of correct answers

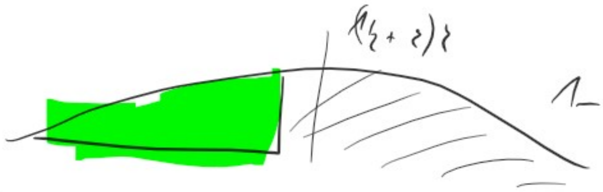
$E(X) = k \cdot (1/2 + \varepsilon)$ $\qquad p = 1/2 + \varepsilon = P_r(X_i = 1)$

$E(X) = k \cdot (1/2 + \varepsilon)$ $\qquad p = 1/2 + \varepsilon = Pr(X_i = 1)$



$Pr(X \geq \frac{k}{2})$

$$\boxed{Pr(X \leq \frac{k}{2})}$$



$1 - Pr(X \leq \frac{k}{2}) = Pr(X \geq \frac{k}{2})$

$$Pr(X \leq (1-\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}}$$

$$(1+\delta)\mu = \frac{k}{2}$$

$$\delta = \frac{\varepsilon}{1/2+\varepsilon} = \frac{\varepsilon}{p}$$

$$Pr\left(X \leq \left(1 - \frac{\varepsilon}{p}\right) \cdot \overbrace{k \cdot p}^{\mu}\right) \leq e^{-\frac{\overbrace{\frac{\varepsilon^2}{p^2}}^{\delta^2} \overbrace{k \cdot p}^{\mu}}{2}} = \dots = e^{-\frac{k \cdot \varepsilon^2}{1+2\varepsilon}}$$

$$e^{-\frac{k \cdot \varepsilon^2}{1+2\varepsilon}} \leq \alpha \qquad / \ln$$

→ desired small error

$$-\frac{k \cdot \varepsilon^2}{1+2\varepsilon} \leq \ln \alpha$$

$$-k\varepsilon^2 \leq \ln \alpha \cdot (1+2\varepsilon)$$

$$\underline{\underline{k \geq \frac{-\ln \alpha \ (1+2\varepsilon)}{\varepsilon_{(n)}^2}}}$$

let n be size of the input

→ if ε does not depend on n (BPP case), then k does not depend on n either

$\rightarrow$ let $\varepsilon = \frac{1}{poly(n)}$

then $k(n)$ is still polynomial in n.

$\rightarrow$ let $\varepsilon = \frac{1}{2^n}$ (possible in PP)

$$k(n) \geq O\left(\frac{\frac{1}{2^n}}{\frac{1}{2^n \cdot 2^n}}\right) = O(2^n)$$