Tutorial 9

29 April 2021 16:04

PROBABILISTIC METHOD 2

1.) Design a randomized experiment in which the desired object is created

2.) if Pr(object with desired properties) > 0

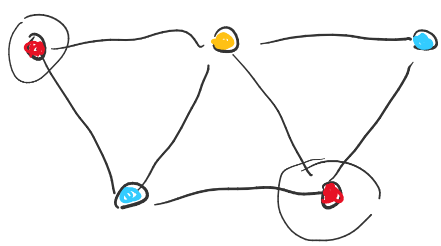
$\Downarrow$

existence of desired object

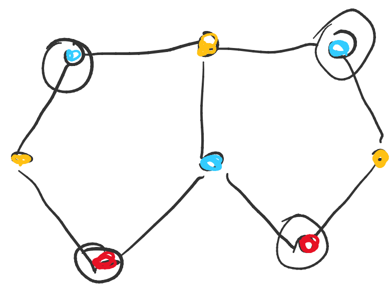
Show existence of graphs with arbitrarily large girth (ℓ) and chromatic number (χ)

$G=(V,E)$ has girth ℓ, if there are no cycles smaller than ℓ.

and has a chromatic number (χ) – smallest number of colors for vertices, such that no edge connects two vertices of the same color.



$\ell = 3$
$\chi = 2$
$d = 2$



$\ell = 5$
$\chi = 3$
$d = 4$

Can you construct a graph with arbitrary girth? ℓ-cycle

Can you construct a graph with arbitrary chromatic number?

K_n

Independence number (α) of graph G is the size of the largest independent set of vertices - without any edges between them

$$\alpha(G) \geq \frac{|V|}{\chi(G)} \qquad \chi(G) \geq \frac{|V|}{\alpha(G)}$$

Intuition:

→ In order to avoid small cycles, the number of edges is small

→ Small number of edges leads to a large independence number

→ large independence number implies small chromatic number

Approach → create a random graph (n-vertices, each of $\binom{n}{2}$ is added with probability $\underline{p}$).

We will show that for sufficiently large n and suitably chosen P, graph with $\ell(G) > a$ and $\chi(G) > b$ exists (gets constructed with probability larger than 0)

We will split this into two events:

1.) The probability that the number of small cycles ($< \ell$) is large ($\frac{n}{2}$) is smaller than 1/2 [E_1 - number of small cycles is larger than $\frac{n}{2}$]

2.) The probability of a large independance set is smaller that 1/2 [E_2 - independance number is "large"]

We want a graph with neither of those properties.

$$\boxed{Pr(E_1) < 1/2 \quad Pr(E_2) < 1/2}$$

$\Downarrow$

$$Pr(\neg E_1 \wedge \neg E_2) = 1 - \underbrace{Pr(E_1 \vee E_2)}_{\text{UNION BOUND}} \geq 1 - Pr(E_1) - Pr(E_2) > 0$$

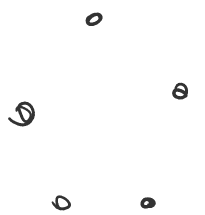
Random graph - add each edge with probability

$p = n^{\lambda - 1}$ $\lambda \in (0, \frac{1}{\ell})$ [importantly $\lambda \cdot \ell < 1$]

We want probability that the number of cycles of size $\leq \ell$ is larger than $\frac{n}{2}$ to be smaller than $1/2$,

X - number of cycles smaller than ℓ

$\Pr(X > \frac{n}{2}) = ?$



$\binom{5}{3}$ possible triangles

but probabilities to get them in random experiment are <u>not</u> independent!

In order to get around the dependence of the cycles we will evaluate $\underline{E(X)}$ instead and use Markov's inequality

$$\Pr(X > t) \leq \frac{E(X)}{t}$$

if we show $\boxed{E(X) < \frac{n}{4}}$ $\overset{\text{by M.I.}}{\Rightarrow}$ $\Pr(X > \frac{n}{2}) < \frac{1}{2}$

In order to calculate $E(X)$ define $N_{x_1, \dots, x_j} = 1$ when vertices $x_1, \dots, x_j$ form a cycle

$= 0$ otherwise

$$X = \sum_{j=3}^{\ell} \sum_{j\text{-tuples}} N_{x_1, \dots, x_j}$$

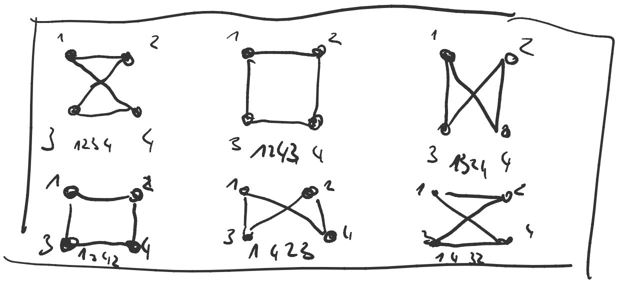
$X = \sum x_i$

$X = \sum \sum \ldots_{n \ldots j} \ldots$
$j=3 \quad j\text{-tuples}$

$\sum_{j=3}^{\ell} \sum_{j\text{-tuples}} E(N_{x_1,\dots,x_j}) \qquad E(X) = \sum_i E(X_i)$

$$E(X) = \sum_{j=3}^{\ell} \sum_{j\text{-tuples}} \Pr(N_{x_1,\dots,x_j} = 1)$$

$$\Pr(N_{x_1,\dots,x_j} = 1) \overset{?}{=} p^j$$



j-tuple has $\frac{(j-1)!}{2}$ unique cycles

$$E(X) \sum_{j=3}^{\ell} \binom{n}{j} \frac{(j-1)!}{2} \cdot p^j \qquad \frac{n!}{(n-j)!\,\cancel{j!}} \cdot \frac{\cancel{(j-1)!}}{2} = \frac{\cancel{n!}}{\cancel{(n-j)!}\,2j}$$

$$< \sum_{j=3}^{\ell} n^j \cdot (n^{\lambda-1})^j \qquad = \frac{n(n-1)\cdots(n-j+1)}{2j} < n^j$$

(for sufficiently large n)

$$= \sum_{j=3}^{\ell} n^j \cdot n^{\lambda j} \cdot n^{-j}$$

$$= \sum_{j=3}^{\ell} n^{\lambda j}$$

$$< \sum_{j=0}^{\ell} (n^\lambda)^j$$

geometric series with gradient n^λ

$$\boxed{\sum_{i=0}^{n} a^i = \frac{1-a^{i+1}}{1-a}}$$

$$= \frac{1-(n^\lambda)^{\ell+1}}{1-(n^\lambda)} = \frac{(n^\lambda)^{\ell+1}-1}{(n^\lambda)-1} < \frac{n^{\lambda\ell} \cdot n^\lambda}{n^\lambda - 1} = \frac{n^{\lambda\ell}}{1-n^{-\lambda}}$$

$$< \frac{n}{4}$$

for sufficiently large n

Let's show that there is n_c s.t. $\forall n > n_c$

$$\frac{n^{\lambda \ell}}{1-n^{-\lambda}} < \frac{n}{c} \qquad \text{(c is an arbitrary positive number)}$$

$$n^{\lambda \ell} < \frac{n}{c} \cdot (1 - n^{-\lambda})$$

$$< \frac{n}{c} - \frac{n^{(1-\lambda)}}{c}$$

$$n^{\lambda \ell} + \frac{n^{(1-\lambda)}}{c} < \frac{n}{c}$$

r.h.s. increases more rapidly assymptotically than l.h.s.

$$\boxed{E(X) < \frac{n}{c_1} \text{ for sufficiently large } n \implies Pr\left(X > \frac{n}{2}\right) < \frac{1}{2}}$$

✓

2.) Independence number $\alpha(G)$ is small

$$Pr(\alpha(G) \geq \underbrace{m}_{\text{specified later}}) < 1/2$$

$$\le \sum_{S \subset V,\ |S| m} Pr(S \text{ is an independent set})$$

$$= \binom{n}{m} (1-p)^{\binom{m}{2}}$$

$$\binom{n}{m} \le n^m$$

$$(1-x) < e^{-x} \Rightarrow (1-p) < e^{-p}$$

$0 < x < 1$

$$< n^m e^{-p \cdot \frac{m(m-1)}{2}}$$

$$m = \left\lceil \frac{3}{p} \cdot \ln(n) \right\rceil$$

$$e^{\ln n} = n$$

$$< n^m n^{-\cancel{p} \cdot \frac{3}{\cancel{p}} \cdot \frac{(m-1)}{2}}$$

$$= n^m n^{-3\frac{(m-1)}{2}}$$

$$= n^{\frac{2m - 3m + 3}{2}}$$

$$= n^{\frac{3-m}{2}} \le n^{3/2 - \frac{3}{2p} \cdot \ln(n)}$$

$$= n^{3/2 - 3/2 \cdot \frac{\ln(n)}{n^{(\lambda-1)}}}$$

$$- \frac{\ln(n)}{n^{-c}} \quad c \in \left(-1, -\frac{\ell-1}{e}\right)$$

$$\approx n$$

$$n \to \infty \quad \not\to 0$$

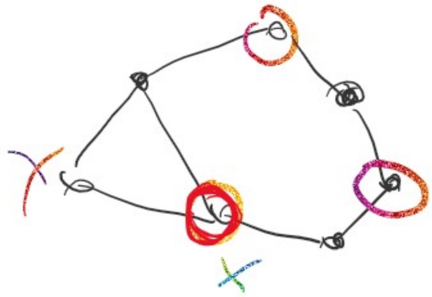
Probability that $\alpha(G) > \lceil \frac{3}{p} \ln(n) \rceil$ is smaller than $1/2$ for sufficiently large n

SUMUP:

The probability to construct a graph G with the number of cycles of size $\leq l$ smaller than $n/2$ and independence number smaller than $\lceil \frac{3}{p} \ln(n) \rceil$ is positive

$\Rightarrow$ IT EXISTS!

From G we can construct G' by deleting a vertex from each small cycle



G' - no cycles $\leq l$

$$\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{n/2}{\frac{3 \cdot n^{(1-\lambda)} \cdot \ln n}{2}} \underset{n \to \infty}{\leadsto} \infty$$

There are graphs with arbitrary girth l and chromatic number k

There are graphs with arbitrary given ℓ and chromatic number χ.