Tutorial 10

06 May 2021 16:10

MARKOV CHAINS

- → basic definitions
- → hitting probabilities
- → hitting time
- → Ergodic theorem (stationary distributions)

Next week

- → Walks on a line
- → 2-SAT (3-SAT)
- → Fair 2-colorability of 3-colorable graphs

Definitions

Markov Chain (MC) is an infinite collection of r.v. $\{X_i\}_{i=0}^{\infty}$

with n outcomes, such that

$x_i \in \{1, \dots, n\}$

$$\forall i \quad \Pr\left(X_i = x_i \mid X_{i-1} = x_{i-1}, X_{i-2} = x_{i-2}, \dots, X_0 = x_0\right)$$

$$\|$$

$$\Pr\left(X_i = x_i \mid X_{i-1} = x_{i-1}\right)$$

NOTE:

generally $\Pr(X_3=3 \mid X_2=2, X_1=1) \neq P(X_3=3 \mid X_2=2, X_1=2)$

INTERPRETATION:

MC`s are the simplest non-trivial stochastic processes.
Each X_i represents a state of the random process in time i
The process can have $\boxed{n}$ states, Markov property says that the state of the process in step $i+1$ depends only on state in time i (and not whole past).

This leads to simplification:

$$\Pr(X_3=3 \mid X_2=2) = \Pr(X_7=3 \mid X_6=2)$$
$$= \Pr(X_{i+1}=3 \mid X_i=2)$$

$= P_{23}$ (probability of moving from state 2 to state 3)

How many probabilities describe MC with n states? $n \times n$

Matrix representation

Transition matrix P

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1n} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2n} \end{pmatrix} \leftarrow \text{sum to 1}$$

$$P = \begin{pmatrix} P_{21} & P_{22} & P_{23} & \dots & P_{2n} \\ \dots & & & & \\ P_{n1} & \dots & & & P_{nn} \end{pmatrix}$$

This matrix stochastic

$$\forall_i \sum_j P_{ij} = 1$$

$$\boxed{\begin{array}{l} \text{if } X_0 = \ell \quad (\text{the process starts in state } \ell) \\ Pr(X_k = m) = ? \end{array}}$$

1.) $\underbrace{(0, 0, \dots 0, \overset{\ell\text{-th position}}{1}, 0 \dots 0)}_{n} \cdot P = (P_{\ell 1}, P_{\ell 2}, \dots, P_{\ell n})$ - (ℓ-th row of P)

2.) $(P_{\ell 1}, P_{\ell 2}, \dots, P_{\ell n}) \cdot P = (0, 0, \dots 0, \overset{\ell\text{th}}{1}, 0, 0 \dots) \cdot P^2$

k.) $(0, 0, \dots, 0, \overset{\ell\text{th}}{1}, 0, \dots, 0) \cdot P^k$ (probability distribution of states in k steps)

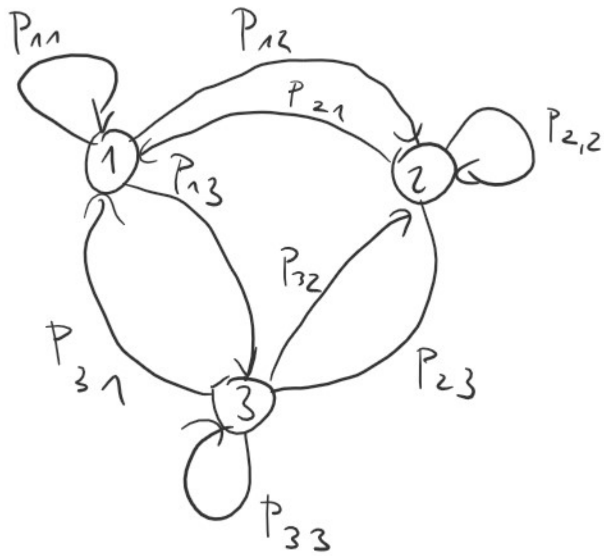
P^k — k-step transition matrix

GRAPH REPRESENTATION

Graph with n vertices (each vertex corresponds to a state) and directed edges labelled by transition probabilities

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \end{pmatrix}$$

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$



Basic properties

$r_{ij}^{(t)} \rightarrow$ the probability to reach j from i in exactly t steps

Hitting probability – the overall probability to reach j from i

$$f_{ij} = \sum_{t=0}^{\infty} r_{ij}^{(t)}$$

Hitting time – the average (expected) time to reach j from i

$$h_{ij} = \sum_{t=0}^{\infty} t \cdot r_{ij}^{(t)}$$

Ergodic theorem (Stationary distribution)

For an ERGODIC MC (all states can reach all other states $w_i \geq 1$

For an ERGODIC MC (all states can reach all other states w.p. ≥ 1 and MC is not periodic), there exists a unique probability vector $\pi = (\pi_1, \dots, \pi_n)$, such that

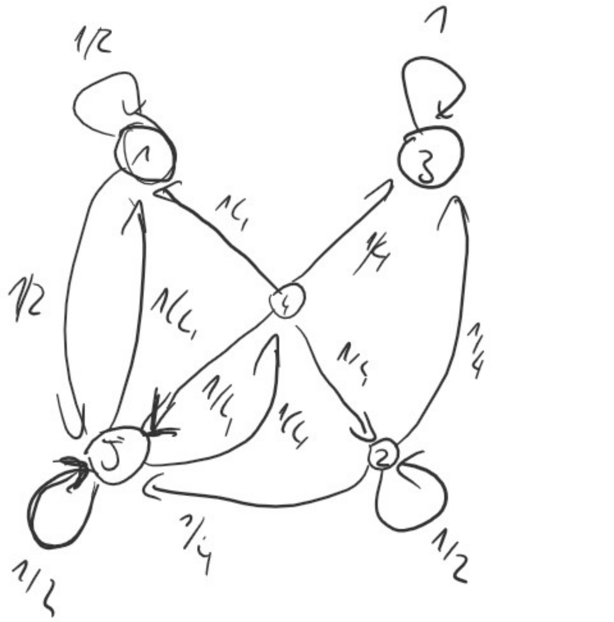
$$\pi P = \pi$$

$\to$ and for all probability vectors p: $\lim_{t \to \infty} p P^t = \pi$.

Periodic MC:

EXERCISES

$$P = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/4 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 & 1/2 \end{pmatrix}$$



TASK: Calculate f_{i4} for each i

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$f_{44} = 1$

$f_{14} = \frac{1}{2} \cdot f_{14} + \frac{1}{2} f_{54} \Rightarrow \underline{f_{14} = f_{54}}$

$f_{24} = \frac{1}{2} \cdot f_{24} + \frac{1}{4} f_{34} + \frac{1}{4} f_{54}$

$f_{34} = f_{34}$ $\quad \boxed{f_{34} = 0}$

$f_{54} = \frac{1}{4} \cdot f_{14} + \frac{1}{4} \cdot f_{44} + \frac{1}{2} f_{54}$

5 equations with 5 variables

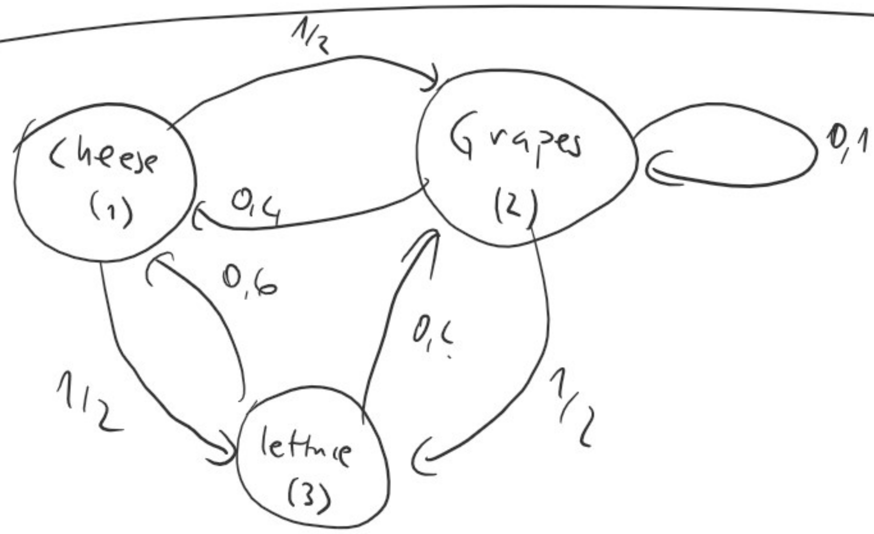
$f_{54} = \frac{1}{4} f_{54} + \frac{1}{4} + \frac{1}{2} f_{54}$

$\frac{1}{4} f_{54} = \frac{1}{4} \Rightarrow f_{54} = 1$ and $f_{14} = 1$

$f_{24} = 0 + \frac{1}{2} f_{24} + \frac{1}{4}$

$f_{24} = \frac{1}{2}$

$\boxed{f_{i4} = (1, \frac{1}{2}, 0, 1, 1)}$



$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0.4 & 0.1 & 0.5 \\ 0.6 & 0.4 & 0 \end{pmatrix}$$

Calculate f_{i3}

$f_{33} = 1$

$f_{23} = 0{,}5\, f_{33} + 0{,}1\, f_{23} + 0{,}4\, f_{13}$

$f_{13} = 0{,}5 f_{33} + 0.5 f_{23} \Rightarrow f_{13} = 1/2\, f_{23} + 1/2$

$f_{23} = 0{,}5 + 0.1\,(f_{23}) + 0.4\left(1/2\, f_{23} + 1/2\right)$

$f_{23} = 1 \Rightarrow f_{13} = 1 \qquad f_{33} = 1$

$f_{i2} = 1 \quad f_{i1} = 1$

$\boxed{\forall_{ij}\ f_{ij} = 1} \Rightarrow \boxed{\text{Ergodic MC}}$

$\boxed{\text{Calculate } h_{i3}}$

$$\boxed{\begin{aligned} h_{33} &= 0 \\ h_{23} &= 1/2\, h_{33} + 1/10 \cdot h_{23} + \tfrac{4}{10} \cdot h_{13} + 1 \\ h_{13} &= 1/2 \cdot h_{33} + 1/2 \cdot h_{23} + 1 \end{aligned}}$$

$: h_{23} = 2 \qquad h_{13} = 2$

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Stationary distribution

$$\pi \cdot P = \underset{\text{eigenvalue}}{c} \cdot \overset{\text{eigen vector}}{\pi}$$

$$\pi \cdot P = \pi$$

$$(\pi_1, \pi_2, \pi_3) \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0.4 & 0.1 & 0.5 \\ 0.6 & 0.4 & 0 \end{pmatrix} = (\pi_1, \pi_2, \pi_3)$$

$$\boxed{\begin{aligned} \pi_1 \cdot 0 + \pi_2 \cdot 0.4 + \pi_3 \cdot 0{,}6 &= \pi_1 \\ \pi_1 \cdot 1/2 + \pi_2 \cdot 1/10 + \pi_3 \cdot \tfrac{2}{5} &= \pi_2 \\ \pi_1 \cdot 1/2 + \pi_2 \cdot 1/2 + \pi_3 \cdot 0 &= \pi_3 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}}$$

$$\pi_1 = \pi_2 = \pi_3 = 1/3$$

$$(1/3, 1/3, 1/3) \cdot P = (1/3, 1/3, 1/3)$$