IA008: Computational Logic

Ehrenfeucht-Fraïssé Games

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Back-and-forth equivalence

Definition

The **quantifier rank** of a formula φ is the nesting depth of quantifiers in φ.

$$\mathrm{qr}(s = t) := 0\,, \qquad \mathrm{qr}(\varphi \wedge \psi) := \max\{\mathrm{qr}(\varphi), \mathrm{qr}(\psi)\}\,,$$
$$\mathrm{qr}(R\bar{t}) := 0\,, \qquad \mathrm{qr}(\varphi \vee \psi) := \max\{\mathrm{qr}(\varphi), \mathrm{qr}(\psi)\}\,,$$
$$\mathrm{qr}(\exists x\varphi) := 1 + \mathrm{qr}(\varphi)\,, \qquad \mathrm{qr}(\neg\varphi) := \mathrm{qr}(\varphi)\,,$$
$$\mathrm{qr}(\forall x\varphi) := 1 + \mathrm{qr}(\varphi)\,.$$

Example

$$\mathrm{qr}\big(\forall x \exists y R(x, y)\big) = 2\,,$$
$$\mathrm{qr}\big(\forall x[P(x) \vee Q(x)] \wedge \forall z[\exists y R(y, z) \vee \exists y R(z, y)]\big) = 2\,.$$

Back-and-forth equivalence

m-equivalence

$$\mathfrak{A}, \bar{a} \equiv_m \mathfrak{B}, \bar{b} \quad \text{:iff} \quad \mathfrak{A} \vDash \varphi(\bar{a}) \iff \mathfrak{B} \vDash \varphi(\bar{b}),$$
$$\text{for all } \varphi(\bar{x}) \text{ with } \operatorname{qr}(\varphi) \le m\,.$$

Lemma

$$\mathfrak{A}, \bar{a} \equiv_{m+1} \mathfrak{B}, \bar{b}$$

if, and only if,

- for all $c \in A$, exists $d \in B$ with $\mathfrak{A}, \bar{a}c \equiv_m \mathfrak{B}, \bar{b}d$ and
- for all $d \in B$, exists $c \in A$ with $\mathfrak{A}, \bar{a}c \equiv_m \mathfrak{B}, \bar{b}d$.

('Back-and-forth conditions')

Back-and-forth equivalence

Proof $(\Leftarrow)$

Suppose $\mathfrak{A} \models \exists x \varphi(\bar{a}, x)$ with $\mathrm{qr}(\varphi) \leq m$.

$\Rightarrow$ exists $c \in A$ with $\mathfrak{A} \models \varphi(\bar{a}, c)$.

By assumption, exists $d \in B$ with $\mathfrak{A}, \bar{a}c \equiv_m \mathfrak{B}, \bar{b}d$.

$\Rightarrow \mathfrak{B} \models \varphi(\bar{b}, d)$

$\Rightarrow \mathfrak{B} \models \exists x \varphi(\bar{b}, x)$

Back-and-forth equivalence

Proof $(\Rightarrow)$

Fix $c \in A$.

$\theta := \{ \psi(\bar{x}, y) \mid \mathrm{qr}(\psi) \leq m\,,\ \mathfrak{A} \models \psi(\bar{a}, c) \}$

$\vartheta := \bigwedge \theta$

$\Rightarrow \mathfrak{A} \models \vartheta(\bar{a}, c)$

$\Rightarrow \mathfrak{A} \models \exists y \vartheta(\bar{a}, y)$

By assumption, $\mathfrak{B} \models \exists y \vartheta(\bar{b}, y)$.

$\Rightarrow$ exists $d \in B$ with $\mathfrak{B} \models \vartheta(\bar{b}, d)$

$\Rightarrow \mathfrak{A}, \bar{a}c \equiv_m \mathfrak{B}, \bar{b}d$

Back-and-forth equivalence

Example linear orders

$$\langle A, \leq \rangle \equiv_m \langle B, \leq \rangle \quad \text{iff} \quad |A| = |B| \quad \text{or} \quad |A|, |B| \geq 2^m - 1 \,.$$

Corollary

There does not exist an FO-formula φ such that

$$\langle A, \leq \rangle \vDash \varphi \quad \text{iff} \quad |A| \text{ is even,}$$

for all finite linear orders.

Ehrenfeucht-Fraïssé Games

Game $\mathcal{G}_m(\mathfrak{A}, \bar{a}; \mathfrak{B}, \bar{b})$

Players: **Spoiler** and **Duplicator**

m **rounds:**

- Spoiler picks an element of one structure.
- Duplicator picks an element of the other structure.

Winning: $\mathfrak{A}, \bar{a}\bar{c} \equiv_0 \mathfrak{B}, \bar{b}\bar{d}$ ($\bar{c} \in A^m$, $\bar{d} \in B^m$ picked elements)

Theorem

$\mathfrak{A}, \bar{a} \equiv_m \mathfrak{B}, \bar{b}$ if, and only if, Duplicator wins $\mathcal{G}_m(\mathfrak{A}, \bar{a}; \mathfrak{B}, \bar{b})$

Words

Word structures

We can represent $u = a_0 \dots a_{n-1} \in \Sigma^*$ as a structure

$$\big\langle \{0, \dots, n-1\}, \leq, (P_c)_{c \in \Sigma} \big\rangle \quad \text{with} \quad P_c := \{\, i < n \mid a_i = c \,\}\,.$$

Lemma

$$u \equiv_m u' \quad \text{and} \quad v \equiv_m v' \quad \Rightarrow \quad uv \equiv_m u'v'\,.$$

Corollary

For $L \subseteq \Sigma^*$ FO-definable, there exists $n \in \mathbb{N}$ such that

$$uv^n w \in L \Leftrightarrow uv^{n+1} w \in L\,, \quad \text{for all } u, v, w \in \Sigma^*.$$

Examples

The following languages are not first-order definable.

- $(aa)^*$
- $\{ a^n b^n \mid n \in \mathbb{N} \}$
- all correctly parenthesised expressions over the alphabet $(\ )\ x$

Monadic Second-Order Logic

Syntax

- element variables: $x, y, z, \dots$
- set variables: $X, Y, Z, \dots$
- atomic formulae: $R(\bar{x})$, $x = y$, $x \in X$
- boolean operations: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
- quantifiers: $\exists x, \forall x, \exists X, \forall X$

Example

- "The set X is empty."

  $\neg \exists x [x \in X]$

- "$X \subseteq Y$"

  $\forall z [z \in X \rightarrow z \in Y]$

- "There exists a path from x to y."

  $\forall Z \big[x \in Z \wedge \forall u \forall v [u \in Z \wedge E(u, v) \rightarrow v \in Z] \rightarrow y \in Z \big]$

Back-and-Forth Equivalence

m-equivalence

$$\mathfrak{A}, \bar{P}, \bar{a} \equiv_m^{\mathrm{MSO}} \mathfrak{B}, \bar{Q}, \bar{b} \quad :\text{iff} \quad \mathfrak{A} \vDash \varphi(\bar{P}, \bar{a}) \Leftrightarrow \mathfrak{B} \vDash \varphi(\bar{Q}, \bar{b})$$

$$\text{for all } \varphi(\bar{X}, \bar{x}) \text{ with } \mathrm{qr}(\varphi) \leq m\,.$$

Ehrenfeucht-Fraïssé Game

Both players choose an **element** or a **set** in each round.

Lemma

$$u \equiv_m^{\mathrm{MSO}} u' \quad \text{and} \quad v \equiv_m^{\mathrm{MSO}} v' \quad \Rightarrow \quad uv \equiv_m^{\mathrm{MSO}} u'v'\,.$$

Automata

Given φ of quantifier rank m, construct $\mathcal{A}_\varphi = \langle Q, \Sigma, \delta, q_0, F \rangle$

Idea: Given $w = a_0 \cdots a_{n-1}$ compute $\equiv_m^{\mathrm{MSO}}$-classes

$$[\varepsilon]_m,\ [a_0]_m,\ [a_0 a_1]_m,\ \dots,\ [a_0 a_1 \cdots a_{n-1}]_m\,.$$

- $Q := \Sigma^* / \equiv_m^{\mathrm{MSO}}$
- $q_0 := [\varepsilon]_m$
- $\delta([w]_m, c) := [wc]_m$
- $F := \{\, [w]_m \mid w \vDash \varphi \,\}$

Theorem

$\mathcal{A}_\varphi$ accepts a word $w \in \Sigma^*$ if, and only if, $w \vDash \varphi$.

Corollary

φ is satisfiable (by a finite word) if, and only if, $\mathcal{A}_\varphi$ accepts some word.

Automata

Given $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ construct $\varphi_{\mathcal{A}}$.

$$\varphi_{\mathcal{A}} := \exists (Z_q)_{q \in Q} \big[\mathrm{ADM} \wedge \mathrm{INIT} \wedge \mathrm{TRANS} \wedge \mathrm{ACC} \big]$$

$$\mathrm{ADM} := \forall x \bigvee_{q \in Q} Z_q$$

$$\mathrm{ACC} := \bigvee_{q \in F} [\mathrm{last} \in Z_q]$$

$$\mathrm{INIT} := \bigvee_{c \in \Sigma} [P_c(\mathrm{first}) \wedge \mathrm{first} \in Z_{\delta(q_0, c)}]$$

Automata

Given $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ construct $\varphi_{\mathcal{A}}$.

$$\varphi_{\mathcal{A}} := \exists (Z_q)_{q \in Q} \big[\mathrm{ADM} \wedge \mathrm{INIT} \wedge \mathrm{TRANS} \wedge \mathrm{ACC} \big]$$

$$\mathrm{TRANS} := \forall x \forall y \big[y = x + 1 \rightarrow \bigwedge_{c \in \Sigma} \bigwedge_{q \in Q} [x \in Z_q \wedge P_c(y) \rightarrow y \in Z_{\delta(q,c)}] \big] .$$

Theorem

$w \vDash \varphi_{\mathcal{A}}$ if, and only if, $\mathcal{A}$ accepts w.

Corollary

A language $L \subseteq \Sigma^*$ is regular if, and only if, it is MSO-definable.

Automata

Corollary

A language $L \subseteq \Sigma^*$ is regular if, and only if, it is MSO-definable.

Example

$L = \{ a^n b^n \mid n \in \mathbb{N} \}$ is not regular.

Proof

Suppose that $\varphi \in \mathrm{MSO}$ defines L.

Set $m := \mathrm{qr}(\varphi)$.

There exist $i < k$ with $a^i \equiv_m^{\mathrm{MSO}} a^k$.

$a^i b^i \in L$

$\Rightarrow a^i b^i \vDash \varphi$

$\Rightarrow a^k b^i \vDash \varphi$

$\Rightarrow a^k b^i \in L$ Contradiction.

The Theorem of Gaifman

Gaifman graph

$\mathcal{G}(\mathfrak{A}) := \langle A, E \rangle$ where $E := \{ \langle c_i, c_j \rangle \mid \bar{c} \in R, \; c_i \neq c_j \}$

$d(x, y)$ distance in $\mathcal{G}(\mathfrak{A})$

Relativisation $\psi^{(r)}(x)$

replace $\exists y \vartheta$ by $\exists y[d(x,y) < r \wedge \vartheta]$

replace $\forall y \vartheta$ by $\forall y[d(x,y) < r \rightarrow \vartheta]$

Basic local sentence

$$\varphi = \exists x_0 \dots x_{n-1} \Big[\bigwedge_{i \neq j} d(x_i, x_j) \geq 2r \wedge \bigwedge_{i<n} \psi^{(r)}(x_i) \Big]$$

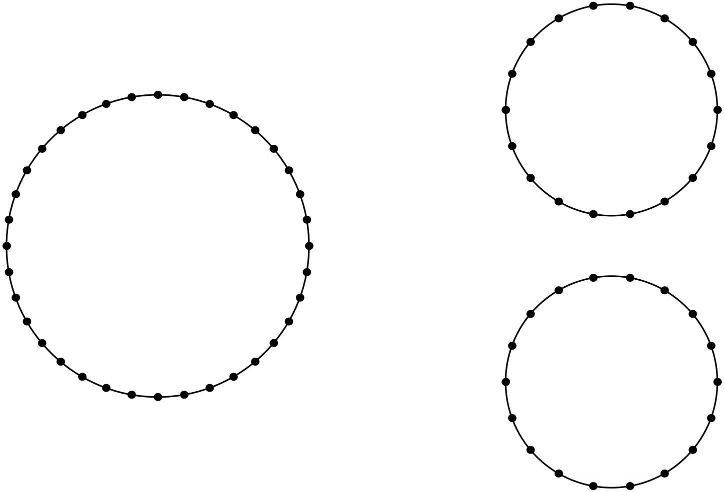
Theorem

Every FO-formula $\varphi(\bar{x})$ is equivalent to a boolean combination of

- basic local sentences and
- formulae of the form $\psi^{(r)}(x_i)$.

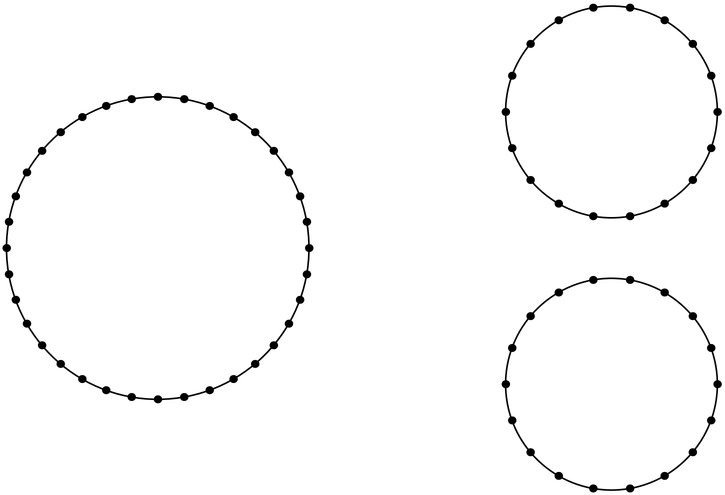
Examples

Connectivity is not first-order definable.



Examples

Planarity is not first-order definable.



Proof

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same basic local sentences up to qr r.

$$N(\bar{a}, r) := \{\, c \mid d(c, a_i) < r \text{ for some } i \,\}$$

Claim $N(\bar{a}, 7^r), \bar{a} \equiv_{g(r)} N(\bar{b}, 7^r), \bar{b} \quad \Rightarrow \quad \mathfrak{A}, \bar{a} \equiv_r \mathfrak{B}, \bar{b}$

Let $r := \mathrm{qr}(\varphi)$.

Claim $\varphi(\bar{x}) \equiv \bigvee \{\, \chi_{\mathfrak{A},\bar{a}}(\bar{x}) \mid \mathfrak{A} \vDash \varphi(\bar{a}) \,\}$,

$$\chi_{\mathfrak{A},\bar{a}}(\bar{x}) := \bigwedge \theta_{\mathfrak{A}} \wedge \bigwedge \theta'_{\mathfrak{A},\bar{a}}(\bar{x}),$$

$$\theta_{\mathfrak{A}} := \Big\{\, \psi \;\Big|\; \psi \text{ basic local},\ \mathrm{qr}(\psi) < h(r),\ \mathfrak{A} \vDash \psi \,\Big\},$$

$$\theta'_{\mathfrak{A}} := \Big\{\, \psi^{(7^r)}(\bar{x}) \;\Big|\; \mathrm{qr}(\psi) < g(r),\ \mathfrak{A} \vDash \psi(\bar{a}) \,\Big\}.$$

$\mathfrak{B} \vDash \varphi(\bar{b})$

$\Rightarrow \mathfrak{B} \vDash \chi_{\mathfrak{B},\bar{b}}(\bar{b})$

Proof

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same basic local sentences up to qr r.

$$N(\bar{a}, r) := \{\, c \mid d(c, a_i) < r \text{ for some } i \,\}$$

Claim $\quad N(\bar{a}, 7^r), \bar{a} \equiv_{g(r)} N(\bar{b}, 7^r), \bar{b} \quad \Rightarrow \quad \mathfrak{A}, \bar{a} \equiv_r \mathfrak{B}, \bar{b}$

Let $r := \mathrm{qr}(\varphi)$.

Claim $\quad \varphi(\bar{x}) \equiv \bigvee \{\, \chi_{\mathfrak{A},\bar{a}}(\bar{x}) \mid \mathfrak{A} \vDash \varphi(\bar{a}) \,\}$,

$$\chi_{\mathfrak{A},\bar{a}}(\bar{x}) := \bigwedge \theta_{\mathfrak{A}} \wedge \bigwedge \theta'_{\mathfrak{A},\bar{a}}(\bar{x}),$$

$$\theta_{\mathfrak{A}} := \big\{\, \psi \;\big|\; \psi \text{ basic local},\ \mathrm{qr}(\psi) < h(r),\ \mathfrak{A} \vDash \psi \,\big\},$$

$$\theta'_{\mathfrak{A}} := \big\{\, \psi^{(7^r)}(\bar{x}) \;\big|\; \mathrm{qr}(\psi) < g(r),\ \mathfrak{A} \vDash \psi(\bar{a}) \,\big\}.$$

$\mathfrak{B} \vDash \chi_{\mathfrak{A},\bar{a}}(\bar{b})$

$\Rightarrow$ $\mathfrak{B}$ satisfies the same basic local sentences as $\mathfrak{A}$ and
$N(\bar{b}, 7^r), \bar{b} \equiv_{g(r)} N(\bar{a}, 7^r), \bar{a}$

$\Rightarrow \mathfrak{B}, \bar{b} \equiv_r \mathfrak{A}, \bar{a}$

$\Rightarrow \mathfrak{B} \vDash \varphi(\bar{b})$

Proof

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same basic local sentences up to qr r.

$$N(\bar{a}, r) := \{\, c \mid d(c, a_i) < r \text{ for some } i \,\}$$

Claim $N(\bar{a}, 7^r), \bar{a} \equiv_{g(r)} N(\bar{b}, 7^r), \bar{b} \quad \Rightarrow \quad \mathfrak{A}, \bar{a} \equiv_r \mathfrak{B}, \bar{b}$

Claim There exists $\psi_{\bar{a},r,m}(\bar{x})$ such that

$$\mathfrak{B} \vDash \psi_{\bar{a},r,m}(\bar{b}) \quad \text{iff} \quad N(\bar{b}, r), \bar{b} \equiv_m N(\bar{a}, r), \bar{a}$$

Set

$$\psi_{\bar{a},r,m} := \bigwedge \theta$$

where

$$\theta := \left\{\, \vartheta^{(r)}(\bar{x}) \;\middle|\; \mathrm{qr}(\vartheta) \leq m \,,\ N(\bar{a}, r) \vDash \vartheta(\bar{a}) \,\right\}$$

Proof

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same basic local sentences up to qr r.

$N(\bar{a}, r) := \{\, c \mid d(c, a_i) < r \text{ for some } i \,\}$

Claim $N(\bar{a}, 7^r), \bar{a} \equiv_{g(r)} N(\bar{b}, 7^r), \bar{b} \quad \Rightarrow \quad \mathfrak{A}, \bar{a} \equiv_r \mathfrak{B}, \bar{b}$

Induction on r

$(r = 0)$ $N(\bar{a}, 1) \equiv_0 N(\bar{b}, 1) \;\Rightarrow\; \mathfrak{A}, \bar{a} \equiv_0 \mathfrak{B}, \bar{b}$

Proof

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same basic local sentences up to qr r.

$$N(\bar{a}, r) := \{\, c \mid d(c, a_i) < r \text{ for some } i \,\}$$

Claim $N(\bar{a}, 7^r), \bar{a} \equiv_{g(r)} N(\bar{b}, 7^r), \bar{b} \quad \Rightarrow \quad \mathfrak{A}, \bar{a} \equiv_r \mathfrak{B}, \bar{b}$

Induction on r

$(r+1)$ Fix $c \in A$.

Case 1 $c \in N(\bar{a}, 2 \cdot 7^r)$

Then $N(c, 7^r) \subseteq N(\bar{a}, 7^{r+1})$ and

$$\begin{aligned} & N(\bar{a}, 7^{r+1}), \bar{a} && \equiv_{g(r)+k+m+1} N(\bar{b}, 7^{r+1}), \bar{b} \\ \Rightarrow\ & N(\bar{a}, 7^{r+1}), \bar{a}c && \equiv_{g(r)+m} N(\bar{b}, 7^{r+1}), \bar{b}d \quad \text{for some } d \in N(\bar{b}, 2 \cdot 7^r), \\ \Rightarrow\ & N(\bar{a}c, 7^r), \bar{a}c && \equiv_{g(r)} N(\bar{b}d, 7^r), \bar{b}d. \end{aligned}$$

$g(r+1) \geq g(r) + k + m + 1$ where k, m are the quantifier-ranks of the formulae defining $N(\bar{a}, 2 \cdot 7^r)$ and $N(\bar{a}c, 7^r)$.

Proof

Case 2 $c \notin N(\bar{a}, 2 \cdot 7^r)$

$$\vartheta_k(\bar{x}) := \bigwedge_{i \neq j} d(x_i, x_j) \geq 4 \cdot 7^r \wedge \bigwedge_i [N(x_i, 7^r) \equiv_{g(r)} N(c, 7^r)] .$$

Let k be maximal such that $N(\bar{a}, 2 \cdot 7^r)$ contains k elements $\bar{c}'$ with

$$N(\bar{a}, 7^{r+1}) \vDash \vartheta_k(\bar{c}') .$$

(Note that $k \leq |\bar{a}|$.)

$\Rightarrow$ k is also the maximum for $N(\bar{b}, 7^{r+1})$

Case 2 a $\mathfrak{B} \vDash \exists \bar{x} \vartheta_{k+1}(\bar{x})$

Then there is some $d \notin N(\bar{b}, 2 \cdot 7^r)$ with

$$N(d, 7^r) \equiv_{g(r)} N(c, 7^r) .$$

Proof

Case 2 $c \notin N(\bar{a}, 2 \cdot 7^r)$

$$\vartheta_k(\bar{x}) := \bigwedge_{i \neq j} d(x_i, x_j) \geq 4 \cdot 7^r \wedge \bigwedge_i [N(x_i, 7^r) \equiv_{g(r)} N(c, 7^r)] .$$

Case 2 b $\mathfrak{B} \models \neg \exists \bar{x} \vartheta_{k+1}(\bar{x})$

$\Rightarrow \mathfrak{A} \models \neg \exists \bar{x} \vartheta_{k+1}(\bar{x})$

$\Rightarrow c \in N(\bar{a}, 7^{r+1})$ satisfies $2 \cdot 7^r \leq d(c, a_i) < 6 \cdot 7^r$

$\Rightarrow$ There is some $d \in N(\bar{b}, 7^{r+1})$ such that

$$2 \cdot 7^r \leq d(d, b_i) < 6 \cdot 7^r \quad \text{and} \quad N(d, 7^r) \equiv_{g(r)} N(c, 7^r) .$$