IA008: Computational Logic

7. Many-Valued Logics

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Basic Concepts

Many-valued logics: Motivation

To some sentences we cannot – or do not want to – assign a truth value since

- they make **presuppositions** that are not fulfilled
  John regrets beating his wife.
  John does not regret beating his wife.
- they refer to **non-existing** objects
  The king of Paris has a pet lion.
- they are too **vague**
  The next supermarket is far away.
- we have **insufficient information**
  The favourite colour of Odysseus was blue.
- we cannot determine their truth
  The Goldbach conjecture holds.

This leads to logics with **truth values** other than 'true' and 'false'.

3-valued logic

truth values 'false' $\bot$, 'uncertain' u, and 'true' $\top$.

A	$\neg A$
$\bot$	$\top$
u	u
$\top$	$\bot$

$\wedge$	$\bot$	u	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
u	$\bot$	u	u
$\top$	$\bot$	u	$\top$

$\vee$	$\bot$	u	$\top$
$\bot$	$\bot$	u	$\top$
u	u	u	$\top$
$\top$	$\top$	$\top$	$\top$

Kleene K3

$\rightarrow$	$\bot$	u	$\top$
$\bot$	$\top$	$\top$	$\top$
u	u	u	$\top$
$\top$	$\bot$	u	$\top$

Łukasiewicz L3

$\rightarrow$	$\bot$	u	$\top$
$\bot$	$\top$	$\top$	$\top$
u	u	$\top$	$\top$
$\top$	$\bot$	u	$\top$

Example

A	B	$A \wedge (A \to B)$	$A \wedge (A \to B) \to B$
$\bot$	$\bot$	$\bot$	$\top$
$\bot$	u	$\bot$	$\top$
$\bot$	$\top$	$\bot$	$\top$
u	$\bot$	u	u
u	u	u	$u/\top$
u	$\top$	u	$\top$
$\top$	$\bot$	$\bot$	$\top$
$\top$	u	u	$u/\top$
$\top$	$\top$	$\top$	$\top$

Fuzzy logic

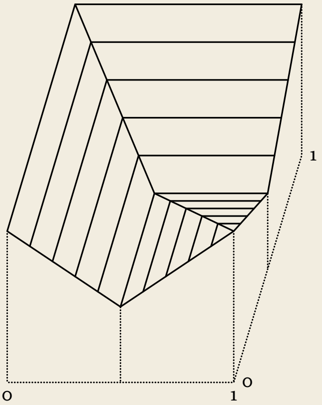
Truth values: $v \in [0,1]$ measuring **how true** a statement is.
0 means ‘false’ and 1 means ‘true’.

Several possible semantics:

$\neg A$	$A \wedge B$	$A \vee B$	$A \to B$
$1 - A$	$A \cdot B$	$1 - (1 - A)(1 - B)$	$1 - A(1 - B)$
$1 - A$	$\min(A, B)$	$\max(A, B)$	$\max(1 - A, B)$
$1 - A$	$\max(A + B - 1, 0)$	$\min(A + B, 1)$	$\min(1 - A + B, 1)$

Example

$$A \wedge (A \rightarrow B) \rightarrow B = \max(1 - \min(A, \max(1 - A, B)), B)$$



Tableaux for L3

statements: $t \leq \varphi$, $\varphi \leq t$, $t \nleq \varphi$, or $\varphi \nleq t$, for $t \in \{\bot, u, \top\}$

Construction

A **tableau** for a formula φ is constructed as follows:

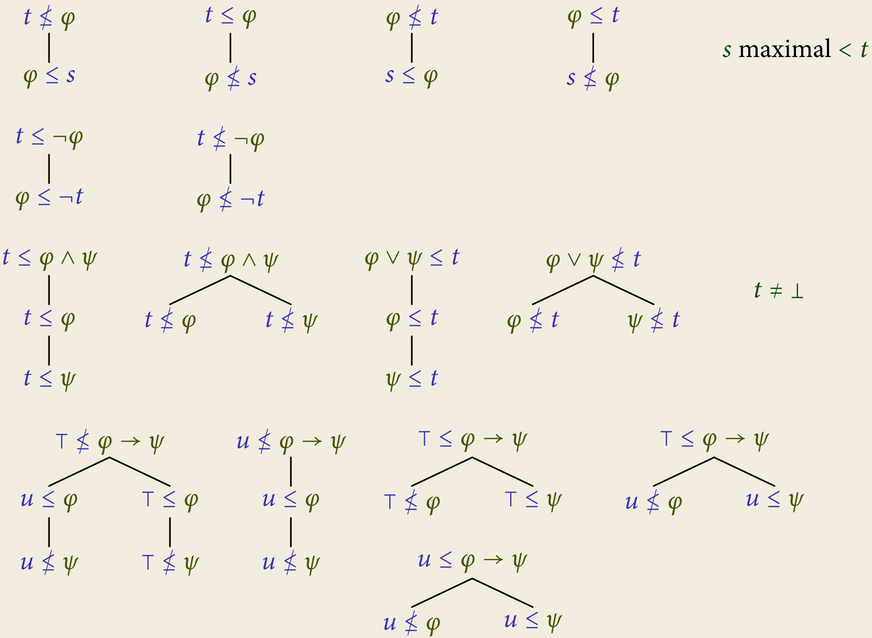
- start with $1 \nleq \varphi$
- choose a branch of the tree
- choose a statement σ on the branch
- choose a rule with head σ
- add it at the bottom of the branch
- repeat until every branch contains one of the following **contradictions**

$$\bot \nleq \varphi \qquad s \leq t \text{ with } s \nleq t \qquad s \leq \varphi \text{ and } t \nleq \varphi \text{ with } t \leq s$$

$$\varphi \nleq \top \qquad s \nleq t \text{ with } s \leq t \qquad \varphi \leq s \text{ and } \varphi \nleq t \text{ with } s \leq t$$

where $s, t \in \{\bot, u, \top\}$ and φ is a formula

Tableaux Rules



$$\begin{array}{c} t \nleq \varphi \\ | \\ \varphi \leq s \end{array} \qquad \begin{array}{c} t \leq \varphi \\ | \\ \varphi \nleq s \end{array} \qquad \begin{array}{c} \varphi \nleq t \\ | \\ s \leq \varphi \end{array} \qquad \begin{array}{c} \varphi \leq t \\ | \\ s \nleq \varphi \end{array}$$

s maximal $< t$

$$\begin{array}{c} t \leq \neg\varphi \\ | \\ \varphi \leq \neg t \end{array} \qquad \begin{array}{c} t \nleq \neg\varphi \\ | \\ \varphi \nleq \neg t \end{array}$$

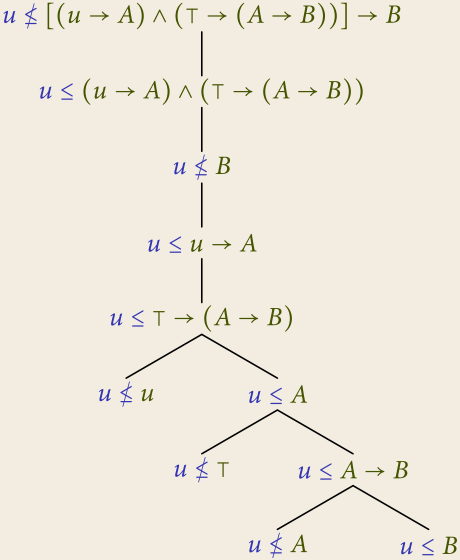
$$\begin{array}{c} t \leq \varphi \wedge \psi \\ | \\ t \leq \varphi \\ | \\ t \leq \psi \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{t \nleq \varphi \wedge \psi} \\ \diagup & \diagdown \\ t \nleq \varphi & t \nleq \psi \end{array} \qquad \begin{array}{c} \varphi \vee \psi \leq t \\ | \\ \varphi \leq t \\ | \\ \psi \leq t \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\varphi \vee \psi \nleq t} \\ \diagup & \diagdown \\ \varphi \nleq t & \psi \nleq t \end{array}$$

$t \neq \bot$

$$\begin{array}{cc} \multicolumn{2}{c}{\top \nleq \varphi \to \psi} \\ \diagup & \diagdown \\ u \leq \varphi & \top \leq \varphi \\ | & | \\ u \nleq \psi & \top \nleq \psi \end{array} \qquad \begin{array}{c} u \nleq \varphi \to \psi \\ | \\ u \leq \varphi \\ | \\ u \nleq \psi \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\top \leq \varphi \to \psi} \\ \diagup & \diagdown \\ \top \nleq \varphi & \top \leq \psi \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\top \leq \varphi \to \psi} \\ \diagup & \diagdown \\ u \nleq \varphi & u \leq \psi \end{array}$$

$$\begin{array}{cc} \multicolumn{2}{c}{u \leq \varphi \to \psi} \\ \diagup & \diagdown \\ u \nleq \varphi & u \leq \psi \end{array}$$

Example



$u \nleq [(u \to A) \wedge (\top \to (A \to B))] \to B$

$u \leq (u \to A) \wedge (\top \to (A \to B))$

$u \nleq B$

$u \leq u \to A$

$u \leq \top \to (A \to B)$

$u \nleq u$ $\quad$ $u \leq A$

$u \nleq \top$ $\quad$ $u \leq A \to B$

$u \nleq A$ $\quad$ $u \leq B$

Intuitionistic Logic

The constructivists view

- We are not interested in **truth** but in **provability.**
- To prove the **existence** of an object is to give a concrete example.

  prove $\exists x \varphi(x) \quad \Leftrightarrow \quad$ find t with $\varphi(t)$

- To prove a **disjunction** is to prove one of the choices.

  prove $\varphi \vee \psi \quad \Leftrightarrow \quad$ prove φ or prove ψ

Goal

A variant of first-order logic that captures these ideas.

Boolean algebras

In **classical logic** the **truth values** form a **boolean algebra** with operations

$$\wedge,\ \vee,\ \neg,\ \top,\ \bot$$

Properties of negation:

$$x \wedge \neg x = \bot \qquad x \vee \neg x = \top$$

Heyting algebras

In **intuitionistic logic** the **truth values** form instead a **Heyting algebra** with operations

$$\wedge,\ \vee,\ \to,\ \top,\ \bot$$

Properties of implication:

$$z \leq x \to y \quad \text{iff} \quad z \wedge x \leq y$$

(that is $x \to y$ is the largest element satisfying $(x \to y) \wedge x \leq y$)

$$x \wedge (x \to y) = x \wedge y \qquad x \to x = \top$$

$$y \wedge (x \to x) = y \qquad x \to (y \wedge z) = (x \to y) \wedge (x \to z)\,.$$

Heyting algebras

In **intuitionistic logic** the **truth values** form instead a **Heyting algebra** with operations

$$\wedge,\ \vee,\ \rightarrow,\ \top,\ \bot$$

Properties of implication:

$$z \leq x \rightarrow y \quad \text{iff} \quad z \wedge x \leq y$$

(that is $x \rightarrow y$ is the largest element satisfying $(x \rightarrow y) \wedge x \leq y$)

$$x \wedge (x \rightarrow y) = x \wedge y \qquad\qquad x \rightarrow x = \top$$

$$y \wedge (x \rightarrow x) = y \qquad\qquad x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)\,.$$

Negation $\neg x := x \rightarrow \bot$

satisfies $x \wedge \neg x = \bot$, but not $x \vee \neg x = \top$

Forcing Frames

Definition

Transition system $\mathfrak{S} = \left\langle S, \leq, (P_i)_{i \in I}, s_0 \right\rangle$ with one edge relation $\leq$ that forms a **partial order**:

- **reflexive** $s \leq s$
- **transitive** $s \leq t \leq u$ implies $s \leq u$
- **anti-symmetric** $s \leq t$ and $t \leq s$ implies $s = t$

The forcing relation

$\mathfrak{S}$ forcing frame, $s \in S$ state, φ formula

$$\begin{aligned}
s \Vdash P_i \quad &:\text{iff} \quad t \in P_i \quad \text{for all } t \geq s \\
s \Vdash \varphi \wedge \psi \quad &:\text{iff} \quad s \Vdash \varphi \text{ and } s \Vdash \psi \\
s \Vdash \varphi \vee \psi \quad &:\text{iff} \quad s \Vdash \varphi \text{ or } s \Vdash \psi \\
s \Vdash \neg\varphi \quad &:\text{iff} \quad t \nVdash \varphi \quad \text{for all } t \geq s \\
s \Vdash \varphi \to \psi \quad &:\text{iff} \quad t \Vdash \varphi \text{ implies } t \Vdash \psi \quad \text{for all } t \geq s
\end{aligned}$$

The **truth value** of φ in $\mathfrak{S}$ is

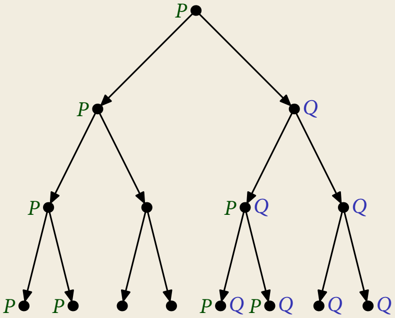
$$[\![\varphi]\!]_{\mathfrak{S}} := \{\, s \in S \mid s \Vdash \varphi \,\},$$

which is **upwards-closed** with respect to $\leq$.

Intuition

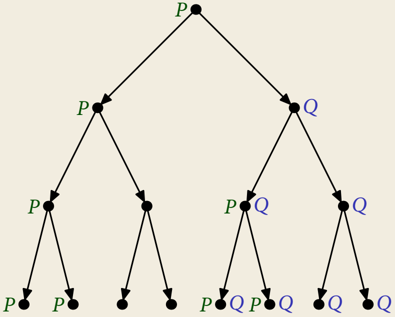
Intuitionistic logic speaks about the **limit behaviour** of φ for large s.

Example



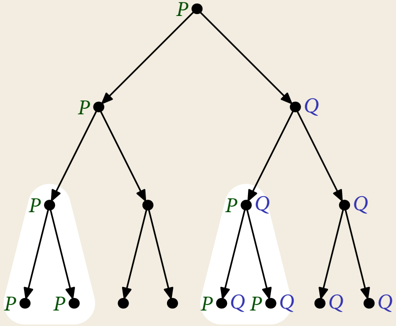
Example

$\varphi := P$



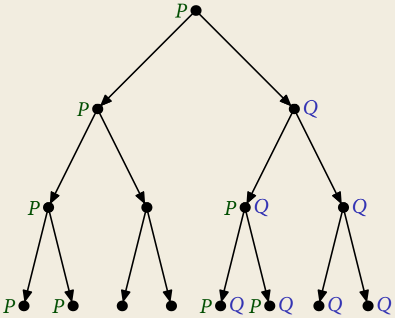
Example

$\varphi := P$



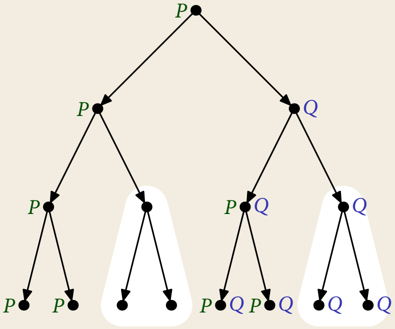
Example

$\varphi := \neg P$



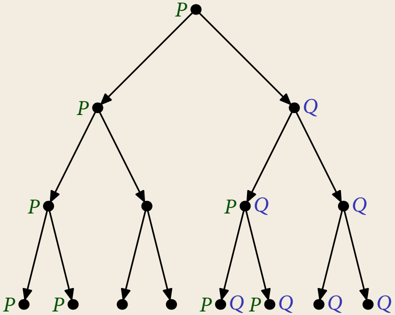
Example

$\varphi := \neg P$



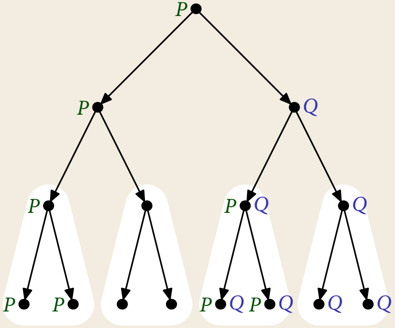
Example

$\varphi := P \vee \neg P$



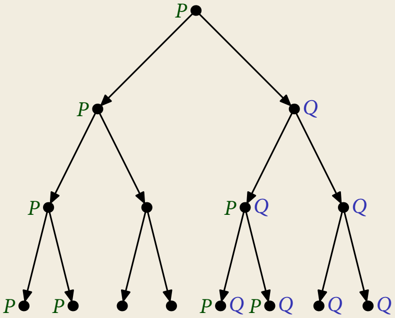
Example

$\varphi := P \vee \neg P$



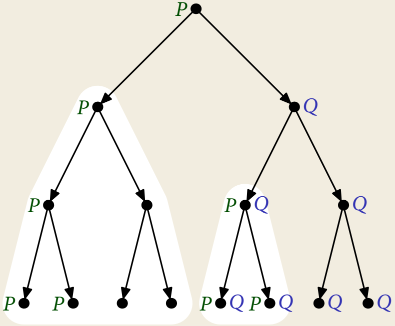
Example

$\varphi := Q \to P$



Example

$\varphi := Q \rightarrow P$



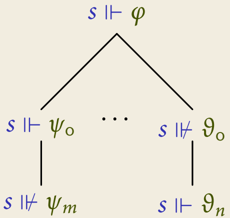
Tableaux for Intuitionistic Logic

Statements

$s \Vdash \varphi \qquad s \nVdash \varphi \qquad s \leq t$

s, t **state labels,** φ a **formula**

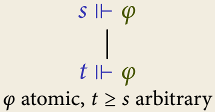
Rules



$s \Vdash \varphi$

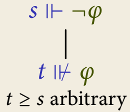
$s \Vdash \psi_0 \quad \dots \quad s \nVdash \vartheta_0$

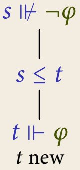
$s \nVdash \psi_m \qquad s \Vdash \vartheta_n$

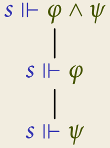


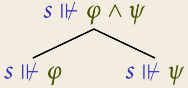
$s \nVdash \varphi$

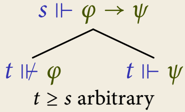
φ atomic

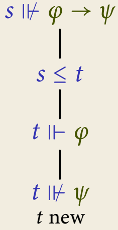


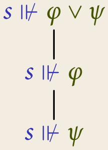


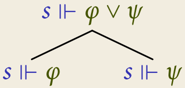


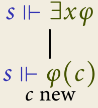


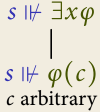


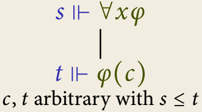


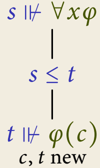












(‘c arbitrary’ means either new or appearing somewhere on the same branch.)

$$s \nVdash A \to (B \to A)$$

$$s \leq t$$

$$t \Vdash A$$

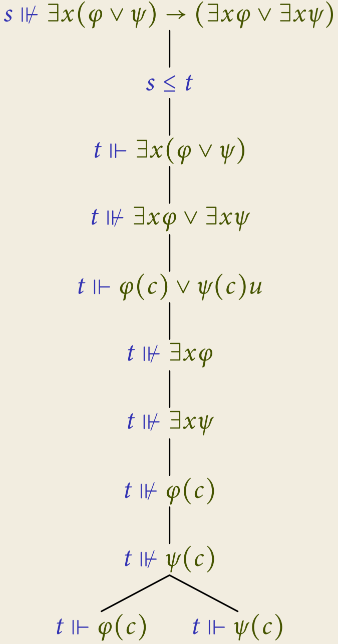
$$t \nVdash B \to A$$

$$t \leq u$$

$$u \Vdash B$$

$$u \nVdash A$$

$$u \Vdash A$$



$s \nVdash \exists x(\varphi \vee \psi) \to (\exists x\varphi \vee \exists x\psi)$

$s \leq t$

$t \Vdash \exists x(\varphi \vee \psi)$

$t \nVdash \exists x\varphi \vee \exists x\psi$

$t \Vdash \varphi(c) \vee \psi(c)u$

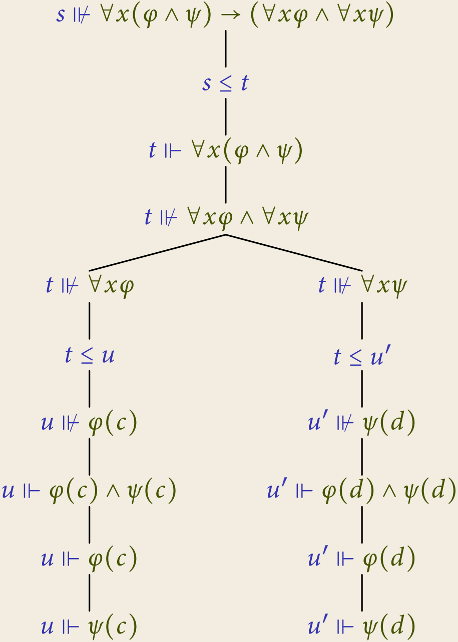
$t \nVdash \exists x\varphi$

$t \nVdash \exists x\psi$

$t \nVdash \varphi(c)$

$t \nVdash \psi(c)$

$t \Vdash \varphi(c)$ $\quad$ $t \Vdash \psi(c)$


$s \nVdash \forall x(\varphi \wedge \psi) \to (\forall x\varphi \wedge \forall x\psi)$
$s \leq t$
$t \Vdash \forall x(\varphi \wedge \psi)$
$t \nVdash \forall x\varphi \wedge \forall x\psi$
$t \nVdash \forall x\varphi$
$t \nVdash \forall x\psi$
$t \leq u$
$t \leq u'$
$u \nVdash \varphi(c)$
$u' \nVdash \psi(d)$
$u \Vdash \varphi(c) \wedge \psi(c)$
$u' \Vdash \varphi(d) \wedge \psi(d)$
$u \Vdash \varphi(c)$
$u' \Vdash \varphi(d)$
$u \Vdash \psi(c)$
$u' \Vdash \psi(d)$