

Exercise 1 We consider words over the alphabet $\{a, b\}$ as transition systems $\langle S, E_s, E_r, P_a, P_b \rangle$ where the states S are the positions, the two predicates P_a and P_b label each position with the corresponding letter, and the two edge relations are

$$E_s = \{ \langle i, i + 1 \rangle \mid i < n - 1 \},$$

$$E_r = \{ \langle i, k \rangle \mid i \leq k < n \}.$$

(where $n = |S|$ is the length of the word). Define the following languages in modal logic.

- (a) All words starting with the letter a .
- (b) All words consisting only of letters a .
- (c) All words ending with the letter a .
- (d) a^*b^*
- (e) All words containing the factor bb .
- (f) All words containing at least two letters b .
- (g) All words containing exactly two letters b .
- (h) $(ab)^*$

Exercise 2 Translate the following formulae into first-order logic.

- (a) $[a]P \rightarrow P$
- (b) $P \rightarrow \langle a \rangle Q$
- (c) $[a](P \wedge \langle b \rangle Q) \rightarrow (\langle a \rangle P \vee \langle b \rangle Q)$

Exercise 3 Prove the following modal formulae using tableaux.

- (a) $\neg \Box \Box P \rightarrow \Diamond \Diamond \neg P$
- (b) $\Box(P \wedge \neg P) \rightarrow \Box Q$
- (c) $\neg \Diamond P \rightarrow \Box(P \rightarrow Q)$
- (d) $\Box(P \leftrightarrow (Q \wedge R)) \rightarrow (\Box P \leftrightarrow (\Box Q \wedge \Box R))$

Prove the following entailment relationships using tableaux.

- (a) $\varphi \rightarrow \Box \varphi \models \Box \varphi \rightarrow \Box \Box \varphi$
- (b) $\forall x \varphi \models \forall x \Box \varphi$

Exercise 4 Find CTL*-formulae defining the following properties of trees with a single predicate P . Which of these statements can be expressed in CTL?

- (a) There is at least one label P .
- (b) Every path contains some P .
- (c) Every path contains at least two P .
- (d) All paths contain infinitely many P .
- (e) Some path contains infinitely many P .

Exercise 5 Express the properties from Exercise 4 in the modal μ -calculus.

Exercise 6 (a) We encode a game graph $\langle V_\diamond, V_\square, E \rangle$ as a transition system $\langle S, E, P_\diamond, P_\square \rangle$ where $S := V_\diamond \cup V_\square$, $P_\diamond := V_\diamond$, and $P_\square := V_\square$. Write a μ -calculus formula stating that the given position is winning for Player \diamond .

(b) (hard) We encode a boolean circuit as a transition system $\langle S, E, P_\wedge, P_\vee, P_\neg, P_o, P_i \rangle$ where P_\wedge, P_\vee , and P_\neg label the three kinds of logic gates, P_o and P_i label the input gates with the corresponding input values, and the output gate is the initial state. Write a μ -calculus formula saying that the output of the circuit is 1.