MARKOV CHAINS - A CRASH COURSE

- → basic definitions
- → hitting probabilities
- → hitting time
- → Ergodic theorem (stationary distribution)

Definitions

A Markov Chain (MC) is an infinite collection of r.v. $\{X_i\}_{i=0}^{\infty}$ with n outcomes, such that

$$\forall_i \quad P_r(X_i = x_i \mid X_{i-1} = x_{i-1}, X_{i-2} = x_2, \ldots, X_0 = x_0)$$
$$\parallel$$
$$P_r(X_i = x_i \mid X_{i-1} = x_{i-1})$$

Note: Generally stochastic processes can depend on the whole past

INTERPRETATION: MC's are the simplest non-trivial stochastic process. Each X_i represents a state of the random process in step in time i. The process can have n states. Markov property says that the state of the process in $i+1^{st}$ step depends only on state in time i and the whole past.

$$P_r(X_3 = 3 \mid X_2 = 2) = P_r(X_7 = 3 \mid X_6 = 2)$$

$= \Pr(X_{i+1} = 3 \mid X_i = 2)$

$= P_{23}$ (probability of moving from state 2 to state 3

How many numbers describe an MC with n states? $n \times n$

Matrix representation - Transition matrix P

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \dots & P_{1n} \\ P_{21} & P_{22} & - & - & P_{2n} \\ & & & & \\ P_{n1} & P_{n2} & \dots & \dots & P_{nn} \end{pmatrix} \leftarrow \text{Sum to } 1$$

The matrix is stochastic

$\forall i \ \sum_j P_{ij} = 1$

$$\boxed{\begin{array}{l} \text{if } X_0 = \ell \quad (\text{the process starts in state } \ell \\ \Pr(X_k = m) = ? \end{array}}$$

1.) $(0, 0, \dots, \underset{\ell\text{-th position}}{1}, \dots 0) \cdot P = (P_{\ell 1}, P_{\ell 2}, \dots, P_{\ell n})$ - (ℓ^{th} row of P)

2.) $(P_{\ell 1}, P_{\ell 2}, \dots, P_{\ell n}) \cdot P = (0, 0, \dots, 1, \dots 0) \cdot P^2$

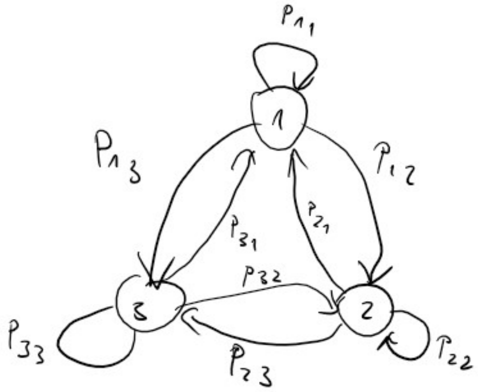
k.) $(0, 0, \dots, 1, \dots 0) \cdot P^k$

P^k - k-step transition matrix

GRAPH REPRESENTATION

Graph with n vertices (vertices correspond to states) and directed edges labelled with transition probabilities

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$



Basic properties

$r_{ij}^{(t)}$ – the probability to reach j from i $\boxed{\text{for the first time}}$ in exactly t steps.

Hitting probabilities – the overall probability to reach j from i

$$f_{ij} = \sum_{t=0}^{\infty} r_{ij}^{(t)}$$

Hitting time – the expected time to reach j from i

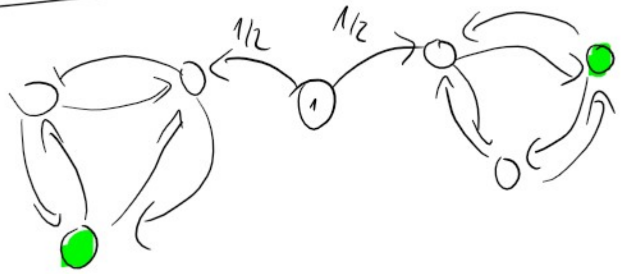
$$h_{ij} = \sum_{t=0}^{\infty} t \cdot r_{ij}^{(t)}$$

Ergodic theorem (stationary distribution)

For an Ergodic MC (all states can reach each other w.p. 1 and MC is not periodic), there exists a unique probability distribution $\pi = (\pi_1, \dots, \pi_n)$, such that $\boxed{\pi \cdot P = \pi}$

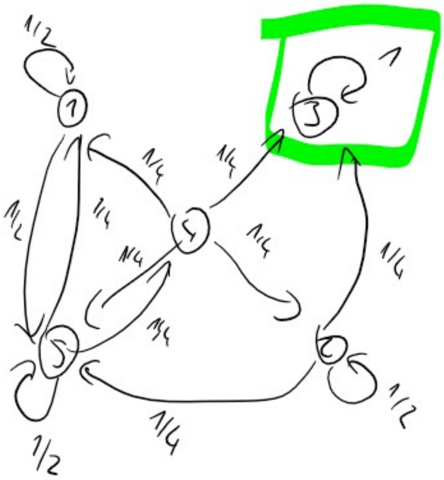
that $\boxed{\pi \cdot P = \pi}$

and for all probability vectors ρ: $\lim_{t \to \infty} \rho \cdot P^t = \pi$



EXERCISES

$$P = \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/4 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 & 1/2 \end{pmatrix} \Longleftrightarrow$$



TASK: Calculate f_{i4} for each i

$f_{4,4} = 1$

$f_{1,4} = \frac{1}{2} \cdot f_{1,4} + \frac{1}{2} f_{5,4} \quad \Rightarrow \quad f_{1,4} = f_{5,4}$

$f_{2,4} = \frac{1}{2} \cdot f_{2,4} + \frac{1}{4} f_{5,4} + \frac{1}{4} f_{3,4}$

$f_{3,4} = 0$

$f \quad \frac{1}{?} f \quad + \frac{1}{4} f \quad + \frac{1}{4} f$

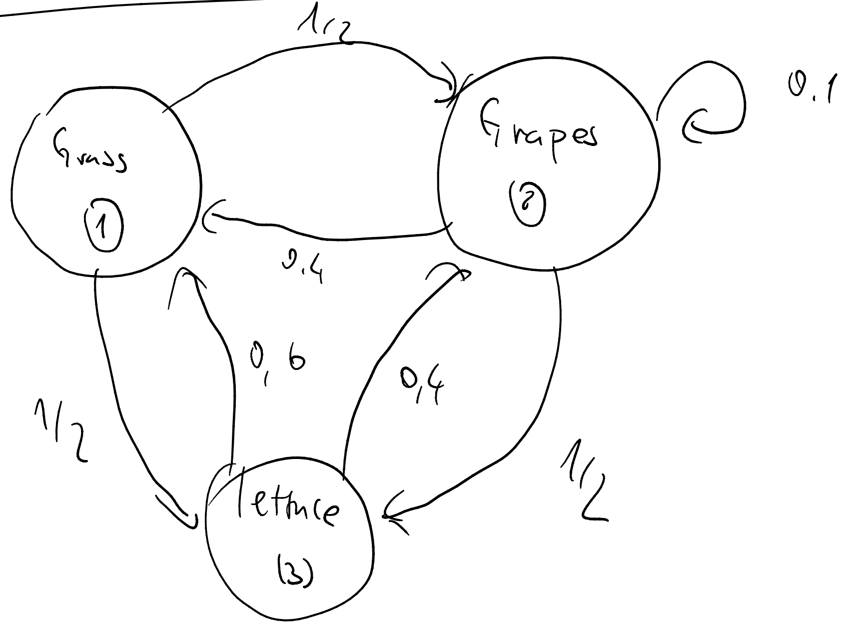
$$f_{5,4} = \tfrac{1}{2} f_{5,4} + \tfrac{1}{4} f_{1,4} + \tfrac{1}{4} f_{4,4}$$

$$f_{5,4} = \tfrac{1}{2} f_{5,4} + \tfrac{1}{4} f_{5,4} + \tfrac{1}{4} \Rightarrow f_{5,4} = 1$$

$$f_{1,4} = 1$$

$$f_{2,4} = \tfrac{1}{2} f_{2,4} + \tfrac{1}{4} + 0 \Rightarrow f_{2,4} = \tfrac{1}{2}$$

$$\boxed{f_{i,4} = (1, \tfrac{1}{2}, 0, 1, 1)}$$



$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0,4 & 0,1 & 0.5 \\ 0.6 & 0.4 & 0 \end{pmatrix}$$

Calculate f_{i3}

$$f_{33} = 1$$

$$f_{23} = \tfrac{1}{2} f_{33} + 0.1 f_{23} + 0.4 f_{13}$$

$$f_{13} = 0,5\, f_{33} + 0.5\, f_{23} \Rightarrow f_{13} = \tfrac{1}{2} f_{2,3} + \tfrac{1}{2}$$

$f_{13} = 0{,}5\, f_{33} + 0.5\, f_{23} \Rightarrow f_{13} = 1/2\, f_{2,3} + 1/2$

$f_{23} = 1/2 + 0.1 \cdot f_{23} + 0.4\, (1/2\, f_{2,3} + 1/2)$

$f_{23} = 1 \Rightarrow f_{13} = 1 \qquad f_{33} = 1$

$\forall_{ij}\, f_{ij} = 1 \Rightarrow$ ERGODIC MC

$$\boxed{\begin{aligned} h_{33} &= 0 \\ h_{23} &= 1/2\, h_{33} + \tfrac{1}{10} h_{23} + \tfrac{4}{10} h_{13} + 1 \\ h_{13} &= 1/2\, h_{33} + 1/2\, h_{23} + 1 \end{aligned}}$$

$h_{23} = 2 \qquad h_{13} = 2$

Stationary distribution

$\Pi \cdot P = \Pi$

$$(\Pi_1, \Pi_2, \Pi_3) \cdot \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0.4 & 0.1 & 0.5 \\ 0.6 & 0.4 & 0 \end{pmatrix} = (\Pi_1, \Pi_2, \Pi_3)$$

$\Pi_1 \cdot 0 + \Pi_2 \cdot 0.4 + \Pi_3 \cdot 0{,}6 = \Pi_1$

$$\pi_1 \cdot 1/2 + \pi_2 \cdot 0.1 + \pi_3 \cdot 0.4 = \pi_2$$

$$\pi_1 \cdot 1/2 + \pi_2 \, 0.5 + \pi_3 \cdot 0 = \pi_3$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\pi_1 = \pi_2 = \pi_3 = 1/3$$

$$(1/3, 1/3, 1/3) \cdot P = (1/3, 1/3, 1/3)$$