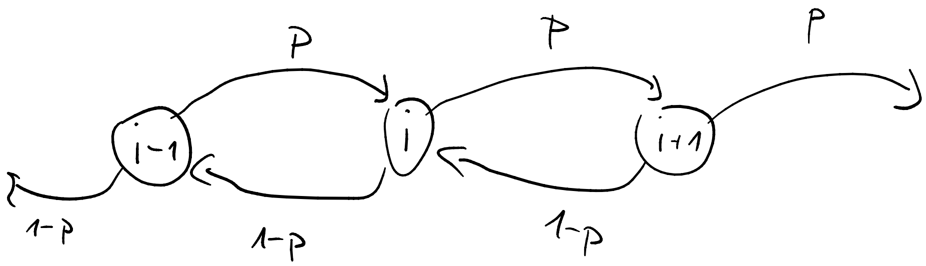
Tutorial 11

05 May 2022 12:02

MARKOV CHAINS II

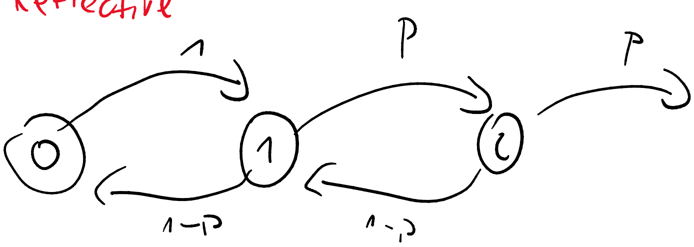
- Walks on a line
- Randomized algorithm for 2-SAT (3-SAT)
- Fair 2-colorability of 3-colorable graphs

Walks on a line

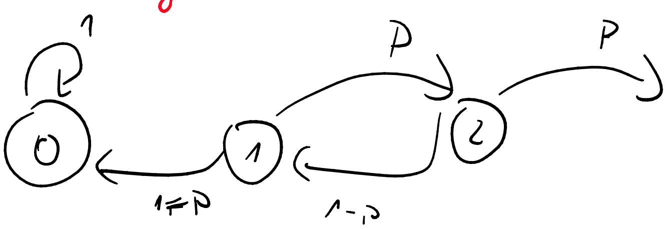


if the line is not infinite in both directions, it contains at least one barrier.

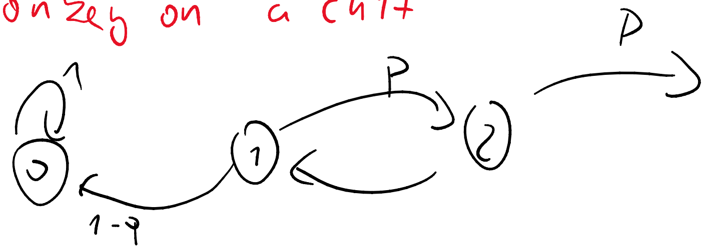
Reflective



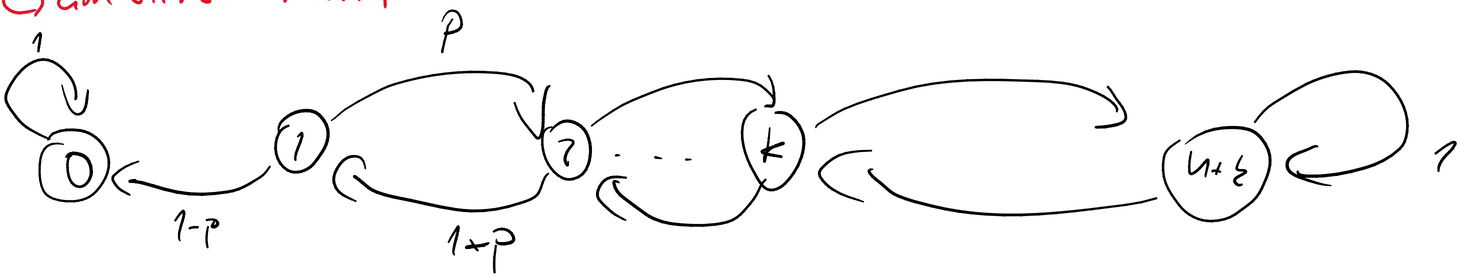
Absorbing



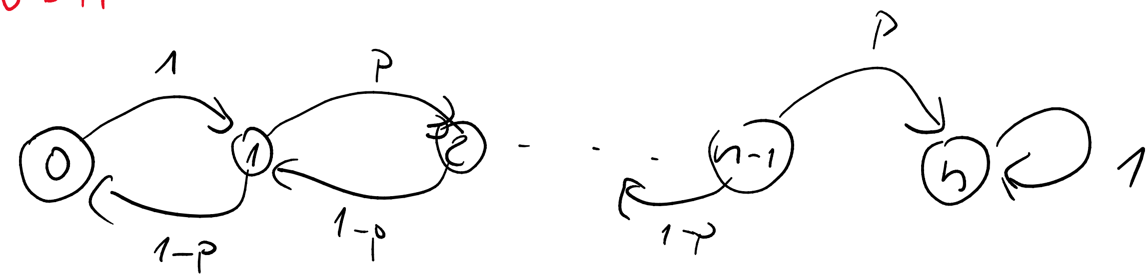
Monkey on a cliff



Gambler's ruin



TODAY



→ What is the expected time to get from 0 to n.

$E_{i,j}$ = expected number of steps to get from i to j

$E_{j,j} = 0$ $\qquad E_{0,n} = E_{0,1} + E_{1,2} + \dots + E_{n-1,n}$

$E_{0,1} = 1$ $\qquad \boxed{\forall i \; E_{i,i+2}} \overset{*}{=} E_{i,i+1} + E_{i+1,i+2}$

$$\forall i \; E_{i,i+1} = 1 + p \underbrace{E_{i+1,i+1}}_{=0} + (1-p) E_{i-1,i+1}$$

$$\forall i \quad E_{i,i+1} = 1 + p\,\boxed{E_{i+1,i+1}} + (1-p)\,E_{i-1,i+1}$$

$$E_{i,i+1} = 1 + (1-p)\left(E_{i-1,i} + E_{i,i+1}\right)$$

$$p\,E_{i,i+1} = 1 + (1-p)\,E_{i-1,i}$$

$$E_{i,i+1} = \frac{1}{p} + \frac{1-p}{p}\,E_{i-1,i}$$

$$E_{i,i+1} = r_i$$

$$r_0 = 1$$
$$r_i = 1/p + \frac{(1-p)}{p}\,r_{i-1}$$

$\rightarrow$ This is called a linear recursive relation.

For given p solutions are easy to find.

Example:

$$p = 1/2$$
$$r_0 = 1 \qquad \Rightarrow r_i = 2i+1$$
$$r_i = 2 + r_{i-1}$$

$$r_0 = c_0$$
$$r_1 = c_1$$
$$r_2 = c_2$$
$$r_{i+3} = a_0 r_0 + a_1 r_1 + a_2 r_2 + C$$

2-SAT

Logical formula with $x_1, \dots, x_n$

$$\overbrace{(x_1 \vee x_2)}^{\text{term}} \wedge (x_3 \vee \neg x_2) \wedge \dots \wedge \overbrace{(\neg x_1 \vee \neg x_5)}^{\text{term}}$$

Is this formula satisfiable? $\rightarrow$ Is there an assignment of truth values (0/1) to variables x_i such that all terms are satisfied?

Randomized procedure:

1.) Assign truth values at random

2.) Find an unsatisfied term (if it does not exist we have a solution) with variables x_a and x_b. Choose at random x_a or x_b and flip its value.

If assignment A exists, how long will it take to find it?
MC counts the number of variables in the intermediate assignment identical to A

$(x_a \vee x_b) \rightarrow$ if this is not satisfied, AT LEAST one of x_a or x_b has a value different from A.

Therefore the probability to get closer to assignment A is AT LEAST $1/2$.

The $\boxed{\text{upper bound}}$ on the expected number of steps is equal to $E_{0,n}$.

$$\boxed{\begin{aligned} & E_{0,1} = 1 \\ & E_{i,i+1} = 1/p + \frac{1-p}{p} E_{i-1,i} \\ & p = 1/2 \\ & E_{i,i+1} = 2 + E_{i-1,i} \\ & E_{i,i+1} = 2i+1 \end{aligned}}$$

$$E_{0,n} = \sum_{i=0}^{n-1} E_{i,i+1} = \sum_{i=0}^{n-1} 2i+1 = n + 2 \cdot \sum_{i=0}^{n-1} i$$

$$= n + 2 \cdot \frac{(n-1)(n-2)}{2} \in O(n^2)$$

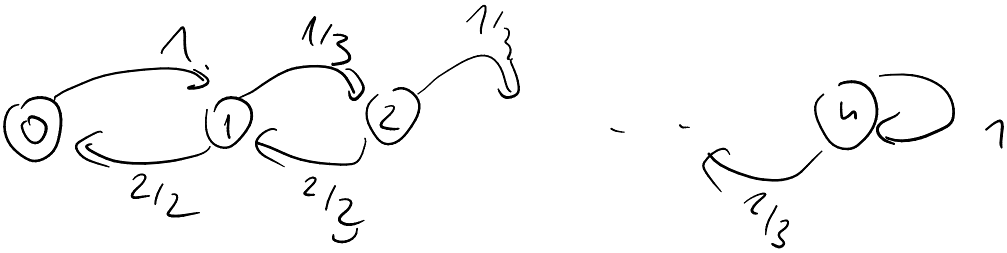
The complete algorithm

Run the procedure $2n^2$ times. If the solution is not found say "I don't know!"

is not found say I don't know.

Why does this fail for 3-SAT?

$(x_a \vee x_b \vee x_c) \to$ terms now have 3 variables



$E_{0,1} = 1$

$E_{i,i+1} = \frac{1}{p} + \frac{1-p}{p} E_{i-1,i}$

for $p = 1/3$

$r_0 = 1$

$\to$

$r_i = 3 + 2r_{i-1}$

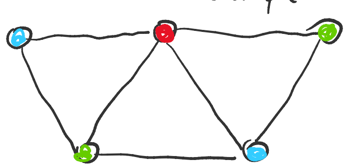
$\boxed{r_i = 2^{i+1} - 3}$

$\Downarrow$

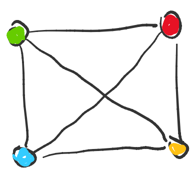
the expected number of steps is exponential

Let G be 3-colorable graph

3 colorable example

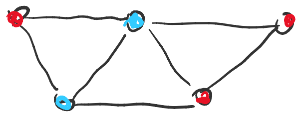


$\to$ not 3-colorable



Task: Find a 2-coloring of G, such that

there is no monochromatic triangle
(fair 2-coloring).



It exists: 3-coloring to fair 2-coloring ?→ Choose one of the colors and change it to one of the remaining 2 colors

→ Choose one of the three colors.

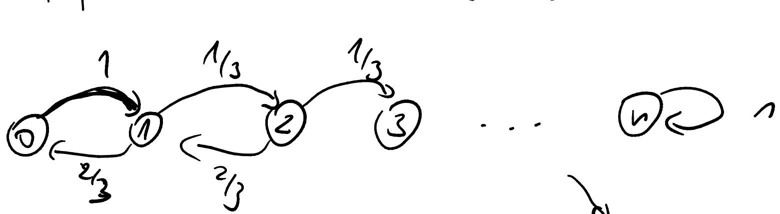
To each node with this color assign one of the two remaining colors at random.

Randomized procedure

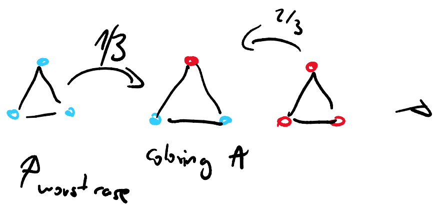
1.) Choose a random 2-coloring

2.) Find a monochromatic triangle → (if there are no such triangle we have a solution)

flip color of one of the nodes chosen at random.

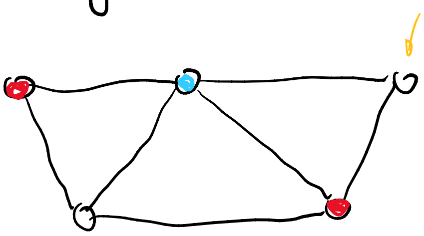


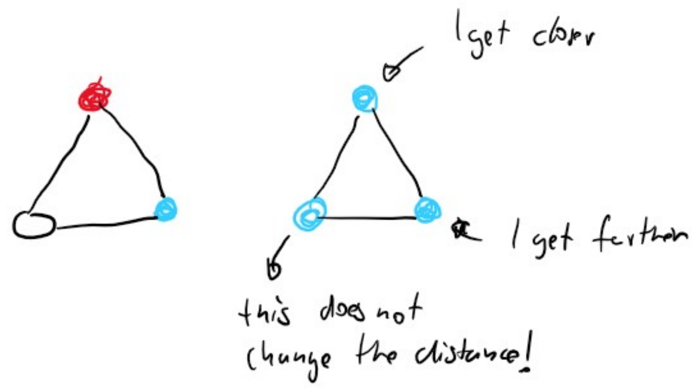
→ Same as 3-SAT!

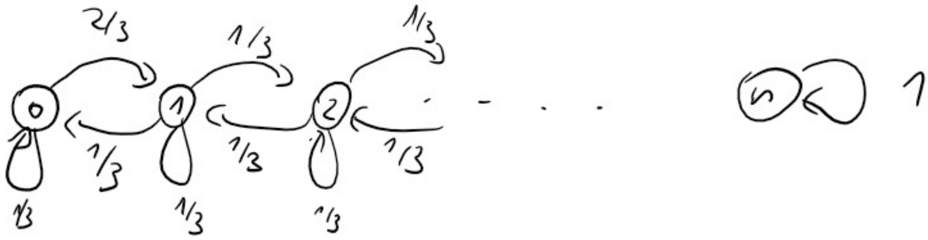


What is important is not the correct 2-coloring itself, but the partial 2-coloring one obtains from the 3-coloring

I don't care about this color







$E_{0,1} = 3/2$

$E_{i,i+1} = 3 + E_{i-1,i}$

$r_0 = 3/2$ $\qquad$ $r_i = 3i + 3/2$

$r_i = 3 + r_{i-1}$

$$E_{op} = \sum_{i=0}^{n-1} E_{i,i+1} = \frac{3n}{2} + 3\sum_{i=0}^{n-1} i \in O(n^2)$$