Tutorial 6

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Basic methods: Moments and deviations

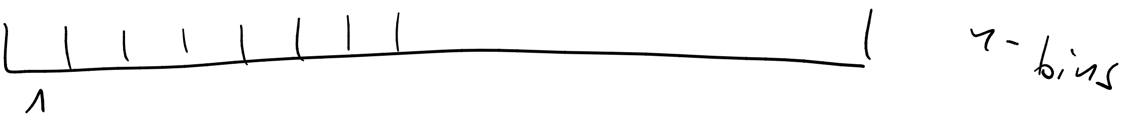
→ Occupancy problem

→ drunken sailor problem

→ coupon collector problem

→ ...

Occupancy problem



m-balls put into bins at random

↓ potentially infinite

Geometric

→ What is the expected number of balls you need to place before 1 of them lands in bin 1

$X = i$ if ith ball is the first one to land in bin 1

$$P_r(X=1) = 1/n$$

$$P(X=2) = \left(\frac{n-1}{\ }\right) \cdot \frac{1}{\ }$$

$$Pr(X=2) = \left(\frac{n-1}{n}\right) \cdot \frac{1}{n}$$

$\vdots$

$$Pr(X=m) = \left(\frac{n-1}{n}\right)^{m-1} \cdot \frac{1}{n}$$

$E(X) = n$ (Probability of success $p = 1/n$) $E(X) = \sum_{i=1}^{\infty} Pr(X=i) \cdot i$

Q: What is the expected number of empty bins if m balls were placed? (Drunken sailors problem)

n^m

$X_i = 1$ if i^{th} bin is empty

$X_i = 0$ if i^{th} bin is occupied

$$Pr(X_i = 1) = \left(\frac{n-1}{n}\right)^m$$

$$E(X_i) = Pr(X_i = 1) = \left(\frac{n-1}{n}\right)^m$$

$$X = \sum_{i=1}^{n} X_i$$

$$E(X) = E\left(\sum_i X_i\right) = \sum_i E(X_i) = \underline{n \cdot \left(\frac{n-1}{n}\right)^m}$$

Q: How many balls do we expect to place in order to fill all the bins?

0 bins occupied

$\Rightarrow$ Probability to fill an unoccupied bin $= 1$ | era 1

1 bin occupied

$\Rightarrow$ Probability to fill an unoccupied bin $= \frac{n-1}{n}$] era 2

$= \frac{n-2}{n}$ } era 3

$n-1$ bins occupied $= 1/n$] era n

For each era X_i - geometric distribution with success probability $p_i = \frac{n-i+1}{n}$

$X = \sum_{i=1}^{n} X_i$ the total number of balls dropped to fill all the bins.

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{n}{n-i+1} = n \sum_{i=1}^{n} \frac{1}{n-i+1}$$

$$E(X) = E\left(\sum_{i=1} X_i\right) = \sum_{i=1} E(X_i) = \sum_{i=1} \frac{n}{n-i+1} = n \sum_{i=1} \frac{1}{n-i+1}$$

$$\underset{\text{reverse order of the sum}}{=} n \cdot \sum_{i=1}^{n} \frac{1}{i} = n \cdot H_n$$ harmonic sum of all elements

$$\approx n \cdot \log n$$

Q: Expected number of bins with k or more balls when n balls were dropped

Pr (j^{th} bin has exactly i balls)

$$= \text{binomial distribution} = \underbrace{\binom{n}{i}}_{\text{which } i \text{ balls}} \underbrace{\left(\frac{1}{n}\right)^i}_{i \text{ successes}} \underbrace{\left(1-\frac{1}{n}\right)^{n-i}}_{n-i \text{ unsuccesses}}$$

$$\leq \binom{n}{i}\left(\frac{1}{n}\right)^i \qquad \left[\binom{n}{i} \leq \left(\frac{ne}{i}\right)^i\right]$$

Euler's number

$$\leq \left(\frac{ne}{i}\right)^i \left(\frac{1}{n}\right)^i = \left(\frac{e}{i}\right)^i$$

$\mathcal{E}_j(k)$ = event that j^{th} bin has k or more balls

$$Pr(\mathcal{E}_j(k)) = \sum_{i=k}^{n} \binom{n}{i}\left(\frac{1}{n}\right)^i \left(\frac{n-1}{n}\right)^{n-i}$$

$$\Pr(\mathcal{E}_j(k)) = \sum_{i=k} \cdots$$

$$\leq \sum_{i=k}^{n} \left(\frac{e}{i}\right)^{i} = \left(\frac{e}{k}\right)^{k} + \left(\frac{e}{k+1}\right)^{k+1} + \dots + \left(\frac{e}{n}\right)^{n}$$

$$\leq \left(\frac{e}{k}\right)^{k} + \left(\frac{e}{k}\right)^{k+1} + \dots + \left(\frac{e}{k}\right)^{n}$$

$$= \left(\frac{e}{k}\right)^{k} \left(\sum_{i=0}^{n} \left(\frac{e}{k}\right)^{i}\right) \qquad \boxed{a<1 \quad \sum_{i=0}^{\infty} a^{i} = \frac{1}{1-a}}$$

$$\left(\frac{e}{k}\right) < 1 \quad \leq \left(\frac{e}{k}\right)^{k} \left(\sum_{i=0}^{\infty} \left(\frac{e}{k}\right)^{i}\right)$$

$$(k \geq 3) \quad = \boxed{\left(\frac{e}{k}\right)^{k} \cdot \frac{1}{1-\frac{e}{k}}}$$

for $k = \left\lceil \frac{e \cdot \ln(n)}{\ln(\ln(n))} \right\rceil$

$$\boxed{\Pr(\mathcal{E}_j(k)) \leq \frac{1}{n^2}}$$

$X_j = 1$ if j^{th} bin has k or more balls

$X_j = 0$ otherwise

$$X = \sum_j X_j$$ (X is the expected number of bins with k or more balls)

$$\boxed{E(X) = \sum_j E(X_j) = \sum_j \Pr \mathcal{E}_j(k)} \leq \frac{1}{n}$$

if $k = \ldots$, $n \to \infty$

Q: What is the probability that at least 1 bin has k or more balls in it?

$$\Pr\left[\bigcup_j \mathcal{E}_j(k)\right] \leq \sum_j \Pr \mathcal{E}_j(k)$$

Mealy's inequality

equality in case events are mutually exclusive

$$\leq 1/n.$$