Tutorial 8

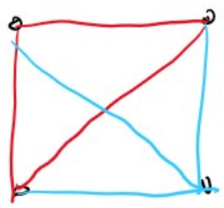
14 April 2022 12:05

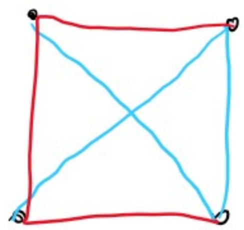
THE PROBABILISTIC METHOD I

Ramsey number

Ramsey number $R(k,t)$ is the smallest n, such that each 2-coloring of the edges of K_n (complete graph of n vertices) has a red subgraph K_k or a blue subgraph K_t.

$\boxed{R(3,3)}$

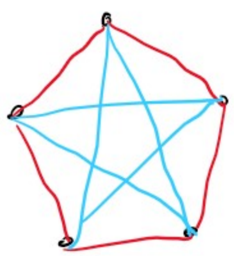




$\Downarrow$

no red or blue triangle $\Rightarrow R(3,3) > 4$

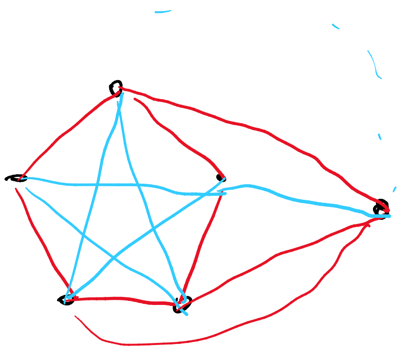
$R(3,3) > 5$



✓

$R(3,3) \overset{?}{>} 6$

$R(3,3) \overset{?}{\geq} 6$



How many colorings of K_6 are there?

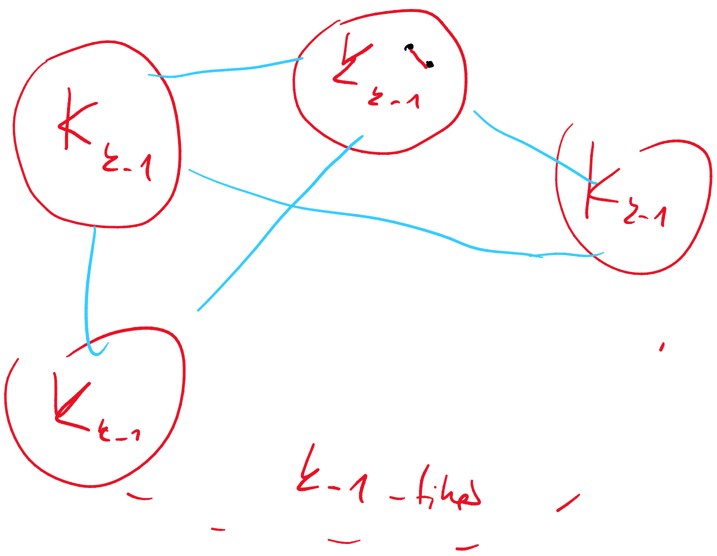
$\binom{6}{2} = 15$

$2^{15} \approx 32000$ colorings

$R(3,3) = 6$

How does $R(k,k)$ scale with k?

Can you find a constructive lower bound?



$R(k,k) > (k-1)^2$

$\boxed{R(3,3) > 4}$

$R(3,5) > 4$

$R(4,4) = 18 > 9$

$43 \leq R(5,5) \leq 49$

16^{V}

PROBABILISTIC ARGUMENT

→ Color each edge at random and if probability of a 'counter example' is larger than 0, the counter example must exist ⇒ lower bound on $R(k,k)$

Thm (from slides)

$$\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1 \Rightarrow R(k,k) > n$$

Let us consider the following random coloring experiment:

Color each edge of K_n red w.p $1/2$

blue w.p. $1/2$

foreach $S \subset V$, $|S| = k$

$r_S = 1$ if graph induced by S is all red

$b_S = 1$ ——— " ——— blue

$$\forall_S \quad Pr(r_S = 1) = \frac{1}{2}^{\binom{k}{2}}$$

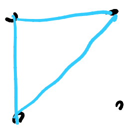
$$\forall_S \quad Pr(b_S = 1) = \frac{1}{2}^{\binom{k}{2}}$$

$$Pr(r_S = 1 \vee b_S = 1) = 2 \cdot \frac{1}{2}^{\binom{k}{2}} = \boxed{2^{1-\binom{k}{2}}}$$

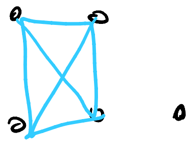
What is the probability that some $S \subset V$ $|S| = k$ is monocolored?

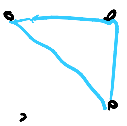
$$\boxed{Pr\left(\bigvee_{S \subset V,\, |S| = k} r_S = 1 \vee b_S = 1\right) < \binom{n}{k} \cdot 2^{1-\binom{k}{2}}}$$

An upper bound to the probability that there is a monocolour K_k in randomly chosen 2-coloring of K_n.

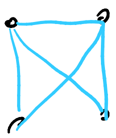


$2^3 = 8$ graphs





$2^3 = 8$ graphs



2 graphs

$$1 - Pr\left(\bigvee_{S \subset V,\, |S| = k} r_S = 1 \vee b_S = 1\right)$$

Probability of randomly creating a 2coloring of K_n without a monochrom K_k.

We need $1 - \Pr\left(\bigvee_{S \subset V, |S| = k} r_S = 1 \vee b_S = 1\right) > 0$

$$\Pr\left(\bigvee_{S \subset V, |S| = k} r_S = 1 \vee b_S = 1\right) < 1$$

$$\leq \binom{n}{k} 2^{1 - \binom{k}{2}} < 1$$

if $n = \lfloor 2^{\frac{k}{2}} \rfloor$ then $R(k,k) \geq n$

Plug $n = \lfloor 2^{\frac{k}{2}} \rfloor$ into and check if it holds

$$\frac{n!}{k!\,\cancel{(n-k)!}} = \overbrace{n \cdot (n-) \cdots (n-k-1)}^{k \text{ - terms}}$$

$$\binom{n}{k} 2^{1 - \binom{k}{2}} \leq \frac{n^k}{k!} \cdot 2^{1 - \frac{k(k-1)}{2}}$$

$$\leq \frac{\left(2^{\frac{k}{2}}\right)^k}{k!} \cdot \frac{2}{2^{\frac{k \cdot (k-1)}{2}}}$$

$$\leq \frac{\cancel{2^{\frac{k^2}{2}}}}{k!} \cdot \frac{2}{\cancel{2^{\frac{k^2}{2}}} \cdot 2^{-\frac{k}{2}}}$$

$$= \frac{2 \cdot 2^{\frac{k}{2}}}{k!} = \frac{2^{1 + \frac{k}{2}}}{k!}$$

$k = 3$ $\quad \frac{2^{3/2}}{3} = \frac{\sqrt{8}}{3} \approx 0.9$

$k = 4$ $\quad \frac{2^{1+2}}{4!} = \frac{8}{24} = 1/3$

$$R(3,3) \geq \lfloor 2^{\frac{3}{2}} \rfloor = \lfloor \sqrt{8} \rfloor = 2$$

$$R(4,4) \geq \lfloor 2^{\frac{4}{2}} \rfloor = 4$$

$$R(8,8) > \lfloor 2^{4} \rfloor = 16$$

$$(k-1)^2 < 2^{\frac{k}{2}}$$

$$k^2 - 2k + 1 < 2^{\frac{k}{2}}$$

$k = 16$

$$2^{\frac{k}{2}} > (k-1)^2$$

$$256 > 225$$

Thm

if $\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1$ for some $\boxed{0 \leq p \leq 1}$

then $R(k,t) > n$

PROOF:

1. Random experiment: With probability p color each edge blue and w.p. $(1-p)$ color each edge red.

$R(4,t)$

$$\binom{n}{4} \cdot p^{6} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1$$

$$\binom{n}{4} \cdot p^6 + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1$$

$$\boxed{p = n^{-2/3}}$$

$$\frac{\cancel{n^4}}{4!} \cdot \cancel{n^{-4}} + \binom{n}{t} \left(1 - n^{-2/3}\right)^{\binom{t}{2}} < 1$$

$$\frac{n^4}{4!} \cdot n^{-4} = \frac{1}{24}$$

$$\binom{n}{t} \left(1 - n^{-2/3}\right)^{\binom{t}{2}} < \frac{23}{24}$$

$$n = \left\lfloor 2^{\frac{t}{2}} \right\rfloor$$???