Tutorial 9

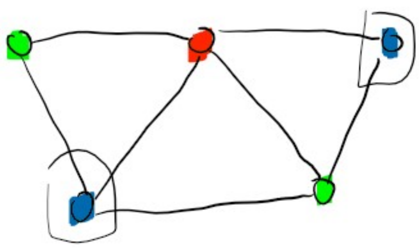
21 April 2022 12:04

# PROBABILISTIC METHOD 2

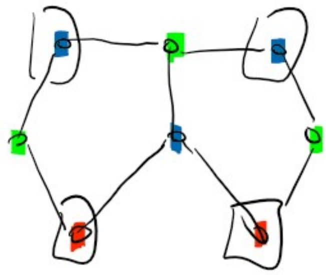
Show existence of graphs with large girth ($\ell$) and chromatic number ($\chi$).

$G=(V,E)$ has girth $\ell$, if there are no cycles smaller than $\ell$.

$G=(V,E)$ has a chromatic number ($\chi$) – smallest number of colors for vertices, such that no edge connects vertices of the same color.



$\ell = 3$
$\chi = 3$
$d = 2$



$\ell = 5$
$\chi = 3$
$\alpha = 4$

→ Can we construct a graph with arbitrary girth $\ell$? $\ell$-cycle

→ Can we construct a graph with arbitrary chromatic number? $K_n$

Independence number ($\alpha$) of a graph $G=(V,E)$ is the size of the largest independent set of vertices. (Vertices without any edges between them:

$\alpha(G) \geq \frac{|V|}{\ldots}$ $\qquad$ $\chi(G) \geq \frac{|V|}{\ldots}$ ✓

$$\alpha(G) \geq \frac{|V|}{\chi(G)} \qquad \chi(G) \geq \frac{|V|}{\alpha(G)} \quad \checkmark$$

Intuition:

→ In order to avoid small cycles, the number of edges should be small.

→ Small number of edges leads to a large independence number.

→ large independence number implies a small chromatic number.

APPROACH: → Create a random graph ($n$-vertices, each of $\binom{n}{2}$ edges w.p. $p$)

Show that for sufficiently large $n$ and suitably chosen $p$, graph with $\ell(G) \geq \ell$ and $\chi(G) \geq k$ exists.

(gets constructed w.p. larger than 0).

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We will split this to two events

1.) Probability that the number of small cycles ($<\ell$) is large ($\frac{n}{2}$) is smaller than $1/2$. [$E_1$ - number of small cycles is large]

2.) The probability of a large independence set is smaller than $1/2$.

[$E_2$ - independence number is large]

$$\boxed{Pr(E_1) < 1/2 \qquad Pr(E_2) < 1/2}$$

$\Downarrow$

$$Pr\{\neg E_1 \wedge \neg E_2\} = 1 - Pr\{E_1 \vee E_2\} \geq 1 - Pr(E_1) - Pr(E_2) > 0$$

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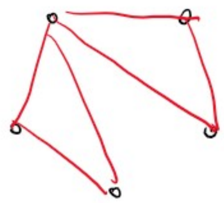
Random graph: add each edge w.p. $p = n^{\lambda - 1}$ $\lambda \in (0, 1/\ell)$

[importantly $\lambda \cdot \ell < 1$]

We want the probability that the number of cycles of size $\leq \ell$ is larger than $\frac{n}{2}$ to be smaller than $1/2$.

$Pr(X > \frac{n}{2})$ $X$ - number of cycles smaller than $\ell$



$\binom{5}{3}$ possible triangles but probabilities to get them in a random experiment is not independent

In order to get around the dependence of cycles we will evaluate $E(X)$ and use Markov's inequality. $Pr(X \geq t) \leq \frac{E(X)}{t}$

if we show $\boxed{E(X) < \frac{n}{4}} \overset{M.I.}{\Longrightarrow} Pr(X > \frac{n}{2}) < 1/2$

In order to evaluate $E(X)$ define r.v.'s $N_{x_1, \ldots, x_j} = 1$

if vertices $x_1, \dots, x_j$ form a cycle

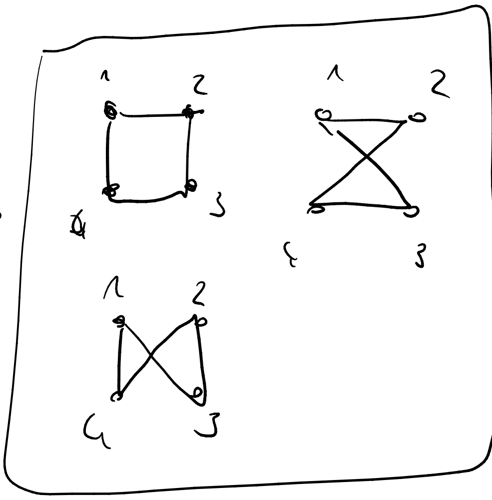
$= 0$ otherwise.

$$X = \sum_{j=3}^{\ell} \sum_{j\text{-tuples}} N_{x_1,\dots,x_j}$$

$$E(X) = \sum_{j=3}^{\ell} \sum_{j\text{-tuples}} \Pr\left(N_{x_1,\dots,x_j} = 1\right)$$

$$= \sum_{j=3}^{\ell} \sum_{j\text{-tuples}} \boxed{\phantom{x}}\, n^j$$

???




j-tuple has ??? cycles $\longrightarrow \frac{j!}{2j} = \frac{(j-1)!}{2}$

$$E(X) = \sum_{j=3}^{\ell} \binom{n}{j} \frac{(j-1)!}{2} \cdot p^j$$

$$\frac{n!}{(n-j)!\,\cancel{j!}} \cdot \frac{\cancel{(j-1)!}}{2} \cdot p^j$$

$$< \sum_{j=3}^{\ell} n^j \cdot (n^{\lambda-1})^j$$

$$\frac{n\cdot(n-1)\cdots(n-j-1)}{2j} < n^j$$

(for sufficiently large $n$)

$$= \sum_{j=3}^{\ell} n^j n^{\lambda j} n^{-j}$$

$$< \sum_{j=0}^{\ell} n^{\lambda j}$$

geometric series with gradient $n^\lambda$

$$\sum_{i=0}^{k} a^i = \frac{1-a^{k+1}}{1-a}$$

choose sufficiently small $\lambda$

$k+1$

$$\sum_{i=0} \qquad \frac{\ }{1-a}$$

sufficiently small lambda for $n^{-\lambda} < 1$

$$= \frac{1-(n^{\lambda})^{l+1}}{1-n^{\lambda}} = \frac{(n^{\lambda})^{l+1}-1}{n^{\lambda}-1} < \frac{n^{\lambda l}\cdot n^{\lambda}}{n^{\lambda}-1} = \boxed{\frac{n^{\lambda l}}{1-n^{-\lambda}}}$$

$$< \frac{n}{4}$$ for sufficiently large $n$

Let's show that there is $n_c$ s.t. $\forall n > n_c$

$$\frac{n^{\lambda l}}{1-n^{-\lambda}} < \frac{n}{c}$$ ( $c$ is an arbitrary positive number)

$$n^{\lambda l} < \frac{n}{c}\cdot\left(1-n^{-\lambda}\right)$$

$\boxed{!\ \lambda l < 1}$ $$< \frac{n}{c} - \frac{n^{(1-\lambda)}}{c}$$

$$n^{\lambda l} + \frac{n^{(1-\lambda)}}{c} < \frac{n}{c}$$ ✓✓

r.h.s is asymptotically increasing faster than l.h.s. Therefore for sufficiently large $n$ their difference is arbitrary

$$\boxed{\begin{array}{l} E(X) < \frac{n}{4} \text{ for sufficiently large } n \text{ (and sufficiently small } \lambda) \\ \Rightarrow \Pr\left(X > \frac{n}{2}\right) < \frac{1}{2}. \end{array}}$$

2.) Independence number $\alpha(G)$ is small.

specified later

$$\Pr(\alpha(G) \geq m) < 1/2$$

$$\leq \sum_{S \subset V,\ |S| = m} \Pr(S \text{ is an independent set})$$

$$= \binom{n}{m} (1-p)^{\binom{m}{2}}$$

$$\binom{n}{m} \leq n^m$$

$$(1-x) < e^{-x}$$

$$0 < x < 1$$

$$< n^m \cdot e^{-p \cdot \frac{m(m-1)}{2}}$$

$$(1-p) < e^{-p}$$

$$m = \left\lceil \frac{3}{p} \cdot \ln(n) \right\rceil$$

$$e^{\ln n} = n$$

$$< n^m \cdot n^{-\cancel{p} \cdot \frac{3}{\cancel{p}} \cdot \frac{(m-1)}{2}}$$

$$= n^m\, n^{-\frac{3(m-1)}{2}}$$

$$= n^{\frac{2m - 3m + 3}{2}}$$

$$= n^{\frac{3-m}{2}} \leq n^{3/2 - \frac{3}{2p}\ln(n)}$$

$$= n^{3/2 - 3/2 \cdot \frac{\ln(n)}{n^{(x-1)}}}$$

$$\approx n^{-\frac{\ln(n)}{n^{-c}}} \qquad c \in \left(-1, \tfrac{1}{e} - 1\right)$$

$$-\ln(n)$$

$-$

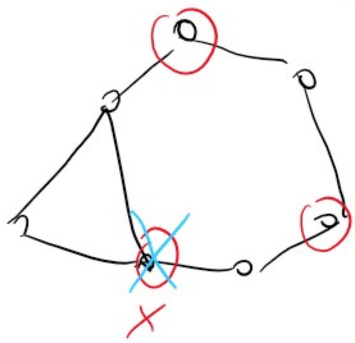
$$\lim_{n \to \infty} n^{-\frac{\ln(n)}{n^{-c}}} = 0$$

Probability that $\alpha(G) > \lceil \frac{3}{p} \ln(n) \rceil$ is smaller than $1/2$ for sufficiently large $n$.

SUMUP: The probability to construct a graph with the number of cycles of size $\leq \ell$ smaller than $n/2$ and independence number smaller than $\lceil \frac{3}{p} \ln(n) \rceil$ is positive $\Rightarrow$ EXISTS

From $G$ we construct $G'$ by deleting a <u>vertex</u> from each small cycle $\Rightarrow$ independence number can only decrease!

$G'$ - no cycles $\leq \ell$



$$\chi(G') > \frac{|V(G')|}{\alpha(G')} > \frac{n/2}{\frac{3 \cdot n^{(1-\lambda)} \cdot \ln n}{2}} \underset{n \to \infty}{\rightsquigarrow} \infty$$