

IAoo8: Computational Logic

5. Ehrenfeucht-Fraïssé Games

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Quantifier rank

Definition

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$$\text{qr}(\forall x\varphi) := 1 + \text{qr}(\varphi).$$

$$\text{qr}(\varphi \wedge \psi) := \max \{ \text{qr}(\varphi), \text{qr}(\psi) \},$$

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Example

$$\text{qr}(\forall x\exists yR(x, y)) = 2,$$

$$\text{qr}(\forall x[P(x) \vee Q(x)] \wedge \forall z[\exists yR(y, z) \vee \exists yR(z, y)]) = 2.$$

Quantifier rank

Lemma

Up to logical equivalence, there are only finitely many formulae of quantifier rank at most m with free variables \bar{x} .

(For a fixed signature Σ that is finite and relational.)

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Back-and-forth equivalence

m -equivalence

$\mathfrak{A}, \bar{a} \equiv_m \mathfrak{B}, \bar{b}$: iff $\mathfrak{A} \models \varphi(\bar{a}) \Leftrightarrow \mathfrak{B} \models \varphi(\bar{b})$,
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Lemma

$\mathfrak{A}, \bar{a} \equiv_{m+1} \mathfrak{B}, \bar{b}$

if, and only if,

- for all $c \in A$, exists $d \in B$ with $\mathfrak{A}, \bar{a}c \equiv_m \mathfrak{B}, \bar{b}d$ and
- for all $d \in B$, exists $c \in A$ with $\mathfrak{A}, \bar{a}c \equiv_m \mathfrak{B}, \bar{b}d$.

(‘back-and-forth conditions’)

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Ehrenfeucht-Fraïssé Games

Game $\mathcal{G}_m(\mathfrak{A}, \bar{a}; \mathfrak{B}, \bar{b})$

Players: Spoiler and Duplicator

m rounds:

- Spoiler picks an element of one structure.
- Duplicator picks an element of the other structure.

Winning: $\mathfrak{A}, \bar{a}\bar{c} \equiv_o \mathfrak{B}, \bar{b}\bar{d}$ ($\bar{c} \in A^m, \bar{d} \in B^m$ picked elements)

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Theorem

$\mathcal{A}, \bar{a} \equiv_m \mathcal{B}, \bar{b}$ if, and only if, Duplicator wins $\mathcal{G}_m(\mathcal{A}, \bar{a}; \mathcal{B}, \bar{b})$

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Example linear orders

$$\langle A, \leq \rangle \equiv_m \langle B, \leq \rangle \quad \text{iff} \quad |A| = |B| \quad \text{or} \quad |A|, |B| \geq 2^m - 1.$$

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For $2^{m-1} \leq k < 2^m$, construct $\varphi_k(x, y)$ with $\text{qr}(\varphi_k) = m$ stating that $x < y$ and there are at least k elements between x and y .

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($m = 0$) trivial

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By inductive hypothesis, Duplicator can then continue the game for $m - 1$ rounds.

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$$\langle A, \leq \rangle \equiv_m \langle B, \leq \rangle \quad \text{iff} \quad |A| = |B| \quad \text{or} \quad |A|, |B| \geq 2^m - 1.$$

Corollary

There does not exist an FO-formula φ such that

$$\langle A, \leq \rangle \models \varphi \quad \text{iff} \quad |A| \text{ is even,}$$

for all finite linear orders.

Words

Word structures

We can represent $u = a_0 \dots a_{n-1} \in \Sigma^*$ as a structure

$$\langle \{0, \dots, n-1\}, \leq, (P_c)_{c \in \Sigma} \rangle \quad \text{with} \quad P_c := \{i < n \mid a_i = c\}.$$

Lemma

$$u \equiv_m u' \quad \text{and} \quad v \equiv_m v' \quad \Rightarrow \quad uv \equiv_m u'v'.$$

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Corollary

For $L \subseteq \Sigma^*$ FO-definable, there exists $n \in \mathbb{N}$ such that

$$uv^n w \in L \Leftrightarrow uv^{n+1} w \in L, \quad \text{for all } u, v, w \in \Sigma^*.$$

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- all correctly parenthesised expressions over the alphabet $() x$

Monadic Second-Order Logic

Syntax

- element variables: x, y, z, \dots
- set variables: X, Y, Z, \dots
- atomic formulae: $R(\bar{x}), x = y, x \in X$
- boolean operations: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
- quantifiers: $\exists x, \forall x, \exists X, \forall X$

Example

- “The set X is empty.”

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- “There exists a path from x to y .”
 $\forall Z [x \in Z \wedge \forall u \forall v [u \in Z \wedge E(u, v) \rightarrow v \in Z] \rightarrow y \in Z]$

Back-and-Forth Equivalence

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$\mathfrak{A}, \bar{P}, \bar{a} \equiv_m^{\text{MSO}} \mathfrak{B}, \bar{Q}, \bar{b}$: iff $\mathfrak{A} \models \varphi(\bar{P}, \bar{a}) \Leftrightarrow \mathfrak{B} \models \varphi(\bar{Q}, \bar{b})$
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Given φ of quantifier rank m , construct $\mathcal{A}_\varphi = \langle Q, \Sigma, \delta, q_0, F \rangle$

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Corollary

φ is satisfiable (by a finite word) if, and only if, \mathcal{A}_φ accepts some word.

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$\Rightarrow a^i b^i \models \varphi$

$\Rightarrow a^k b^i \models \varphi$

$\Rightarrow a^k b^i \in L$ Contradiction.

The Theorem of Gaifman

Gaifman graph

$\mathcal{G}(\mathfrak{A}) := \langle A, E \rangle$ where $E := \{ \langle c_i, c_j \rangle \mid \bar{c} \in R, c_i \neq c_j \}$

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Basic local sentence

$\varphi = \exists x_0 \dots x_{n-1} \left[\bigwedge_{i \neq j} d(x_i, x_j) \geq 2r \wedge \bigwedge_{i < n} \psi^{(r)}(x_i) \right]$

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$$\varphi = \exists x_0 \dots x_{n-1} \left[\bigwedge_{i \neq j} d(x_i, x_j) \geq 2r \wedge \bigwedge_{i < n} \psi^{(r)}(x_i) \right]$$

Theorem

Every FO-formula $\varphi(\bar{x})$ is equivalent to a boolean combination of

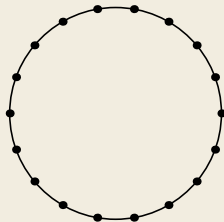
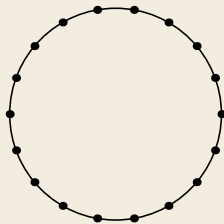
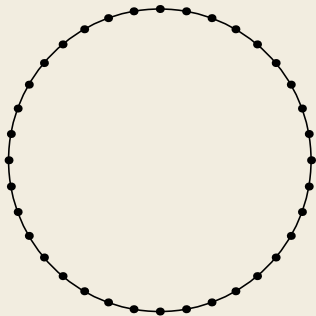
- basic local sentences and
- formulae of the form $\psi^{(r)}(x_i)$.

Examples

Connectivity is not first-order definable.

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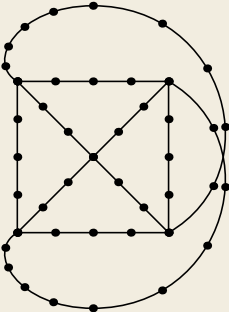
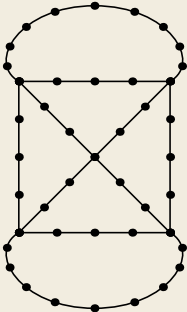


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Proof

Suppose that \mathfrak{A} and \mathfrak{B} satisfy the same basic local sentences up to qr $h(r)$.

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Claim $N(\bar{a}, r), \bar{a} \equiv_{g(r)} N(\bar{b}, r), \bar{b} \Rightarrow \mathfrak{A}, \bar{a} \equiv_r \mathfrak{B}, \bar{b}$

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Using this claim, we can prove the theorem as follows.

Let $r := qr(\varphi)$. It is sufficient to show that

$$\varphi(\bar{x}) \equiv \bigvee \{ \chi_{\mathfrak{A}, \bar{a}}(\bar{x}) \mid \mathfrak{A} \models \varphi(\bar{a}) \},$$

where

$$\chi_{\mathfrak{A}, \bar{a}}(\bar{x}) := \bigwedge \theta_{\mathfrak{A}} \wedge \bigwedge \theta'_{\mathfrak{A}, \bar{a}}(\bar{x}),$$

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Suppose that

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$$\Rightarrow \mathfrak{B} \models \varphi(\bar{b})$$

Proof

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Set

$$\psi_{\bar{a}, r, m} := \bigwedge \theta$$

where

$$\theta := \{ \vartheta^{(r)}(\bar{x}) \mid \text{qr}(\vartheta) \leq m, N(\bar{a}, r) \models \vartheta(\bar{a}) \}$$

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$$(r = 0) N(\bar{a}, 1) \equiv_0 N(\bar{b}, 1) \Rightarrow \mathfrak{A}, \bar{a} \equiv_0 \mathfrak{B}, \bar{b}$$

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Induction on r

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Case 1 $c \in N(\bar{a}, 2 \cdot 7^r)$

(c is close to the \bar{a})

Proof

Suppose that \mathfrak{A} and \mathfrak{B} satisfy the same basic local sentences up to qr $h(r)$.

$$N(\bar{a}, r) := \{ c \mid d(c, a_i) < r \text{ for some } i \}$$

Claim $N(\bar{a}, 7^r), \bar{a} \equiv_{g(r)} N(\bar{b}, 7^r), \bar{b} \Rightarrow \mathfrak{A}, \bar{a} \equiv_r \mathfrak{B}, \bar{b}$

Induction on r

$(r + 1)$ Fix $c \in A$.

Case 1 $c \in N(\bar{a}, 2 \cdot 7^r)$ (c is close to the \bar{a})

Then $N(c, 7^r) \subseteq N(\bar{a}, 7^{r+1})$ and

$$\begin{aligned} & N(\bar{a}, 7^{r+1}), \bar{a} \equiv_{g(r)+k+m+1} N(\bar{b}, 7^{r+1}), \bar{b} \\ \Rightarrow & N(\bar{a}, 7^{r+1}), \bar{a}c \equiv_{g(r)+m} N(\bar{b}, 7^{r+1}), \bar{b}d \quad \text{for some } d \in N(\bar{b}, 2 \cdot 7^r), \\ \Rightarrow & N(\bar{a}c, 7^r), \bar{a}c \equiv_{g(r)} N(\bar{b}d, 7^r), \bar{b}d. \end{aligned}$$

$g(r + 1) \geq g(r) + k + m + 1$ where k, m are the quantifier-ranks of the formulae defining $N(\bar{a}, 2 \cdot 7^r)$ and $N(\bar{a}c, 7^r)$.

Proof

Case 2 $c \notin N(\bar{a}, 2 \cdot 7^r)$

(c is not close to the \bar{a})

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$$\mathfrak{g}_k(\bar{x}) := \bigwedge_{i \neq j} d(x_i, x_j) \geq 4 \cdot 7^r \wedge \bigwedge_i [N(x_i, 7^r) \equiv_{g(r)} N(c, 7^r)].$$

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Let k be maximal such that $N(\bar{a}, 2 \cdot 7^r)$ contains k elements \bar{c}' with

$$N(\bar{a}, 7^{r+1}) \models \mathfrak{g}_k(\bar{c}').$$

(Note that $k \leq |\bar{a}|$.)

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$\Rightarrow k$ is also the maximum for $N(\bar{b}, 7^{r+1})$

Case 2 a $\mathfrak{B} \models \exists \bar{x} \mathfrak{g}_{k+1}(\bar{x})$

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Then there is some $d \notin N(\bar{b}, 2 \cdot 7^r)$ with

$$N(d, 7^r) \equiv_{g(r)} N(c, 7^r).$$

Proof

Case 2 $c \notin N(\bar{a}, 2 \cdot 7^r)$

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Case 2 b $\mathfrak{B} \models \neg \exists \bar{x} \mathfrak{g}_{k+1}(\bar{x})$ (c is at medium distance from the \bar{a})

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$$\Rightarrow \mathfrak{A} \models \neg \exists \bar{x} \mathfrak{g}_{k+1}(\bar{x})$$

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$$\Rightarrow \mathfrak{A} \models \neg \exists \bar{x} \mathfrak{G}_{k+1}(\bar{x})$$

$$\Rightarrow c \in N(\bar{a}, 7^{r+1}) \text{ satisfies } 2 \cdot 7^r \leq d(c, a_i) < 6 \cdot 7^r$$

Proof

Case 2 $c \notin N(\bar{a}, 2 \cdot 7^r)$

$$\mathfrak{g}_k(\bar{x}) := \bigwedge_{i \neq j} d(x_i, x_j) \geq 4 \cdot 7^r \wedge \bigwedge_i [N(x_i, 7^r) \equiv_{g(r)} N(c, 7^r)].$$

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$$\Rightarrow \mathfrak{A} \models \neg \exists \bar{x} \mathfrak{g}_{k+1}(\bar{x})$$

$\Rightarrow c \in N(\bar{a}, 7^{r+1})$ satisfies $2 \cdot 7^r \leq d(c, a_i) < 6 \cdot 7^r$

\Rightarrow There is some $d \in N(\bar{b}, 7^{r+1})$ such that

$$2 \cdot 7^r \leq d(d, b_i) < 6 \cdot 7^r \quad \text{and} \quad N(d, 7^r) \equiv_{g(r)} N(c, 7^r).$$