IA038 Types and Proofs

5. The Normalization Theorem

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Notation:

- $\longrightarrow$ ... one-step reduction, reducing one redex occurence into a contractum within a term
- $\twoheadrightarrow$ ... multi-step reduction, zero to any number of single step reductions
- $=$ means $(\longrightarrow \cup \longrightarrow^{-1})^{\star}$ (or just equality from the denotational semantics)

Uniqueness of the normal form

Reduction $\longrightarrow$ satisfies *Church-Rosser property* ($\longrightarrow$ *is* CR) iff for any M, M_1, M_2 such that $M \twoheadrightarrow M_1$ and $M \twoheadrightarrow M_2$, there is a term M_3 such that $M_1 \twoheadrightarrow M_3$ and $M_2 \twoheadrightarrow M_3$.
Diagramatically, the following diagram comutes:

$$\begin{tikzcd}
M \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\mathbf{CR}"] & M_1 \arrow[d, two heads, dashed] \\
M_2 \arrow[r, two heads, dashed] & M_3
\end{tikzcd}$$

When Church-Rosser property holds true, we have:

When

$$M = N,$$

there exists a term Z such that

$$M \twoheadrightarrow Z \quad \text{a} \quad N \twoheadrightarrow Z.$$

The proof follows from the definition of CR by induction on transitivity definition of $=$:

When $M = N$ follows from $M \twoheadrightarrow N$, we can have $N \equiv Z$.

When $M = N$ follows from $N = L$ and $L = M$, then by induction assumption (IA) there are terms Z_1 and Z_2 such that $N \twoheadrightarrow Z_1$, $L \twoheadrightarrow Z_1$ a $L \twoheadrightarrow Z_2$, $M \twoheadrightarrow Z_2$. Church-Rosser property now yields the existence of Z, such that $Z_1 \twoheadrightarrow Z$ and $Z_2 \twoheadrightarrow Z$.

Proof schema:

$$\begin{tikzcd}
N \arrow[dr] & = \arrow[d, phantom, "\mathrm{IA}"] & L \arrow[dl] \arrow[dr] & = \arrow[d, phantom, "\mathrm{IA}"] & M \arrow[dl] \\
& Z_1 \arrow[dr] & \mathrm{CR} & Z_2 \arrow[dl] & \\
& & Z & &
\end{tikzcd}$$

Hence Church-Rosser property guarantees for normal forms:

reachability For N being a normal form of M such that $M = N$, there is $M \twoheadrightarrow N$
(When $M = N$, there is Z s.t. $N \twoheadrightarrow Z$ and $M \twoheadrightarrow Z$, but N is normal, hence $Z \equiv N$.)

uniqueness For any term, there axists at most one normal form.
(For two normal forms, N_1, N_2, there is $N_1 = N_2$ based on transtivity of $=$; from CR there exists a term Z, for which $N_1 \twoheadrightarrow Z$ and $N_2 \twoheadrightarrow Z$, normality gives $N_1 \equiv Z$ and $N_2 \equiv Z$ and thus $N_1 \equiv N_2$.)

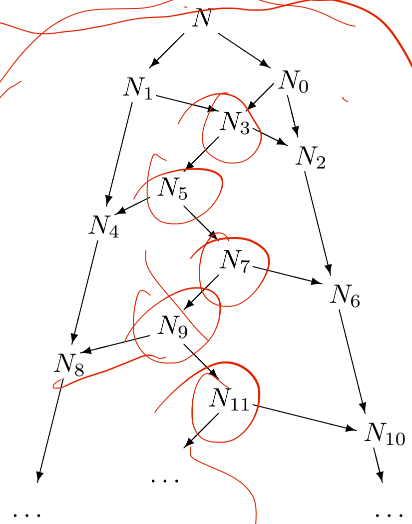
Property: Reduction in the λ-calculus is CR. $((\lambda x.M)N \to M[x/N])$

Proving CR means analyzing redex/contractum behavior and development of redexes dusring subsequent reductions.

Just beware of the fact that the definition of CR *cannot* be simplified to single/step reductions in the assumption:
Weak Church-Rosser property:

$$\begin{tikzcd}
M \arrow[r] \arrow[d] \arrow[dr, phantom, "\textbf{WCR}"] & M_1 \arrow[d, dashed, two heads] \\
M_2 \arrow[r, dashed, two heads] & M_3
\end{tikzcd}$$

CR generally does *not* follow from WCR when the existence of normal forms is not guaranteed: The following is and infinite graph of a reduction which is WCR but not CR:



The weak normalization theorem
Weakly normalizing reduction: for every term there exists a reduction into normal form
Strongly normalizing reduction: Any reduction sequence terminates (with a normal form).

Reduction in the λ-calculus is weakly normalizing.

Proof by Alan Turing (found by Hindley-Seldin in 1980):

Define *redex degree* for a redex

$$(\lambda x^{T_1}.P^{T_2})Q^{T_1}$$

as the number of symbols in the type T_1.

Typed terms can be ordered by the highers redex degree contained in them, or by highest different degree of differing redex in case the highest are equal, or by the length of terms in case all redexes are of the same degree.

Converting (reducing) a redex with the highest order, no other redexes of that order arise. Hence this reduction lowers the degree, yielding a finite maximum length of any reduction sequence converting redexes of the highest degree.

$(\lambda x.M) \ldots x \in M$

Strong normalization proof by Gandy:
Define λI-terms based on λ-terms and interpreted by numerical values and functions.
Typed λI-terms are defined as a subset of λ-terms only allowing to form abstractions when bound variable occurs in the bodz, i.e.
$\lambda x.M$ is a λI-term for M being a λI-term and x a variable only if $x \in \mathrm{FV}(M)$.
Provide a fixed interpretation of types by a function $\mathcal{J}$ assigning $\mathcal{J}(T) \equiv \mathcal{N}$, for any atomic type T.
Use monotonic functions defined as a family H:
For every type T define a structure

$$(H_T, <_T)$$

consisting of a set with an ordering:

$$H_T \subseteq \mathcal{J}(T)$$

$$<_T \subseteq H_T \times H_T$$

as:

$$(H_T, <_T) \equiv (\mathcal{N}, <_{\mathcal{N}}), \text{ with } T \text{ atomic type},$$

$$H_{T_1 \to T_2} \equiv \{\, f \in \mathcal{J}(T_1 \to T_2) \mid \forall a, a' \in H_T : fa \in H_{T_2} \wedge \\ \wedge (a <_{T_1} a' \Rightarrow fa <_{T_2} fa') \,\}$$

$$f <_{T_1 \to T_2} g \text{ iff } \forall a \in H_{T_1} : fa <_{T_2} ga.$$

(The initial ordering $<_{\mathcal{N}}$ it the less-than relation over natural numbers.)

This defines a hierarchy of sets H_T as subsets of T containing just monotonic functions preserving the ordering when applied to arguments).
Property: When λI-terms only use members of H_T, they define monotonic functions of its free variables with members from H_T¿
Let M be a λI-term of type T. For φ an interpretation assigning to free variables of M values from H, then the value

$$\mathcal{E}[\![M]\!]\varphi$$

is monotonic and belongs to H_T.
Proof by induction on the structure of M:

1. For $M \equiv x$ we assumed $\mathcal{E}[\![x]\!]\varphi \in H_T$ is monotonic.
2. For $M \equiv M_1 M_2$, from the inductive assumption it follows that

$$\mathcal{E}[\![M_1]\!]\varphi \in H_{T_1 \to T}$$

and

$$\mathcal{E}[\![M_2]\!]\varphi \in H_{T_1},$$

both monotonic it their free variables.
From the definition of $H_{T_1 \to T}$ it follows

$$(\mathcal{E}[\![M_1]\!]\varphi)(\mathcal{E}[\![M_2]\!]\varphi) \in H_T$$

and this is again monotonic function of the free variables from $M_1 M_2$.

3. For $M \equiv \lambda x.M_1$, if follows from the induction assumption,

$$\mathcal{E}[\![M_1]\!]\varphi' \in H_{T_2}$$

for any φ', which assigns a value from H_{T_1} to variable x of type T. This means

$$\forall a \in H_{T_1} : (\lambda\!\!\lambda \boldsymbol{x} \mathcal{E}[\![M_1]\!][x \leftarrow \boldsymbol{x}]\varphi)a \in H_{T_2}$$

and also (because $\mathcal{E}[\![M_1]\!]\varphi'$ is monotonic function of its free variables and $x \in \mathrm{FV}(M_1)$)

$$\forall a, a' \in H_{T_1} : (\lambda\!\!\lambda \boldsymbol{x} \mathcal{E}[\![M_1]\!][x \leftarrow \boldsymbol{x}]\varphi)a < (\lambda\!\!\lambda \boldsymbol{x} \mathcal{E}[\![M_1]\!][x \leftarrow \boldsymbol{x}]\varphi)a'$$

and so

$$\mathcal{E}[\![\lambda x.M]\!]\varphi \in H_T$$

whioch is again monotonic in its free variables.

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For definition of term norms, define several base functions over natural numbers:

1. For each atomic type T define the following constants:
   0^T of type T,
   S_T of type $T \to T$ a
   $+_T$ of type $T \to T \to T$,
   interpreted so that

$$\begin{aligned}\varphi(0^T) &= 0,\\ \varphi(S_T) &= \text{successor function } \mathcal{N} \to \mathcal{N} \text{ a}\\ \varphi(+_T) &= \text{addition function } \mathcal{N} \to \mathcal{N} \to \mathcal{N}\ (\equiv \mathcal{N} \times \mathcal{N} \to \mathcal{N}).\end{aligned}$$

   Applications of contants $+_T$ will be written in infix notation,

$$M +_T N$$

   instead of

$$+_T MN.$$

2. For any types T_1, T_2 define terms $S_{T_1 \to T_2}$, $+_{T_1 \to T_2}$ as

$$S_{T_1 \to T_2} \equiv \lambda f^{T_1 \to T_2} x^{T_1}.S_{T_2}(fx)$$

   and

$$+_{T_1 \to T_2} \equiv \lambda f^{T_1 \to T_2} g^{T_1 \to T_2} x^{T_1}.(fx) +_{T_2} (gx).$$

3. Let T be an atomic type, T_1, T_2 general types. Define a family of typed terms L as

   (a) $L^T \equiv 0^T$

   (b) $L^{T \to T} \equiv \lambda x^T . x$

   (c) $L^{(T_1 \to T_2) \to T} \equiv \lambda f^{T_1 \to T_2} . L^{T_2 \to T} . f L^{T_1}$

   (d) $L^{T_1 \to (T_2 \to T_3)} \equiv \lambda x^{T_1} y^{T_2} . (L^{T_1 \to T_3} x) +_{T_3} (L^{T_2 \to T_3} y)$

In particular, we have now the constant $0 \in \mathcal{N}$, ther successor function $\mathcal{N} \to \mathcal{N}$ and the addition function $\mathcal{N} \times \mathcal{N} \to \mathcal{N}$, plus an extension of those in all types together with functions $L^{T_1 \to T}$, with T atomic, allowing to project those into natural numbers.
Each of these functions is monotonic and belonging into H:
For any interpretation φ and any type T the following can be easily proved:

1. $\mathcal{E}[\![0^T]\!]\varphi \in H^T$,

2. $\mathcal{E}[\![S_T]\!]\varphi \in H^{T \to T}$,

3. $\mathcal{E}[\![+_T]\!]\varphi \in H^{T \to T \to T}$,

4. $\mathcal{E}[\![L^T]\!]\varphi \in H^T$.

Now define a transformation $|| \; ||$ mapping terms of any tzpe into a base type (and thus to natural numbers).

In order to define $|| \; ||$, general λ-terms have to be embedded into λI-terms using $\star$ transformation: For any term M define λI-term $M^\star$ as:

1. $x^\star \equiv x$, for x a variable.
2. $(MN)^\star \equiv M^\star N^\star$.
3. $(\lambda x^{T_1}.M^{T_2})^\star \equiv \lambda x.S_{T_1}(M^\star + L^{T_1 \to T_2} x)$.

A key property: Let M be a redex and N the corresponding contractum, i.e.

$$M \longrightarrow\!\!\!\!\!\rightarrow N.$$

Then it holds true

$$\mathcal{E}[\![M^\star]\!]\varphi > \mathcal{E}[\![N^\star]\!]\varphi,$$

for any interpretation φ.
For proof, take

$$M \equiv (\lambda x.P)Q$$

and

$$N \equiv P[Q/x].$$

Then

$$\begin{aligned} M^\star &\equiv (\lambda x.S(P^\star + Lx))Q^\star \\ &\equiv (S(P^\star + Lx))[Q^\star/x] \\ &\equiv S(P^\star[Q^\star/x] + LQ^\star) \end{aligned}$$

and

$$\begin{aligned} N^\star &\equiv (P[Q/x])^\star \\ &\equiv P^\star[Q^\star/x]. \end{aligned}$$

The result follows from monotonicity of λI-terms.

Monoonicity of reductions
Let M, N be terms satisfying

$$M \twoheadrightarrow N.$$

Then it holds true

$$\mathcal{E}[\![M^\star]\!]\varphi > \mathcal{E}[\![N^\star]\!]\varphi.$$

Norms of general terms

Transformation $||\ ||$ can now be defined as:
For any term M of type T_1 define $||M||$ as

$$||M|| \equiv \mathcal{E}[\![L^{T_1 \to T}(M^\star)]\!]\varphi,$$

where T is atomic.
Hence for M, N terms satisfying

$$M \longrightarrow N.$$

it holds true that

$$||M|| > ||N||.$$

(By monotonicity of L and the previous result.)

Strong normalization of typed λ-terms:
The typed λ-calculus reduction is strongly normalizing.

No infinite sequence of reductions

$$M \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow \cdots$$

may exist for any term M because of the monotonicity property: The value of $||M||$ is finite, and decreases after any reduction

$$M \longrightarrow N$$

so that

$$||M|| > ||N||.$$

There is no decreasing infinite sequence of natural numbers starting with $||M||$, so also no infinite sequence of reductions.

The value of $||M||$ is uniquely defined for any interpretation φ because of CR.
The upper bound for the length of any sequence of reductions starting with M can be given as $||M||$.