Smullyan signed tableaux system and the Sequent Calculus

formulas prefixed by T or F correspond to the position on the left or on the right in the sequents:

$$\frac{\mathrm{T}\ \neg X}{\mathrm{F}\ X} \quad \frac{\mathrm{F}\ \neg X}{\mathrm{T}\ X} \qquad \frac{\mathrm{T}\ X \wedge Y}{\begin{array}{c}\mathrm{T}\ X\\ \mathrm{T}\ Y\end{array}} \quad \frac{\mathrm{F}\ X \wedge Y}{\mathrm{F}\ X \mid \mathrm{F}\ Y} \qquad \frac{\mathrm{T}\ X \vee Y}{\mathrm{T}\ X \mid \mathrm{T}\ Y} \quad \frac{\mathrm{F}\ X \vee Y}{\begin{array}{c}\mathrm{F}\ X\\ \mathrm{F}\ Y\end{array}} \qquad \frac{\mathrm{T}\ X \to Y}{\mathrm{F}\ X \mid \mathrm{T}\ Y} \quad \frac{\mathrm{F}\ X \to Y}{\begin{array}{c}\mathrm{T}\ X\\ \mathrm{F}\ Y\end{array}}$$

Example:

$F(\neg(A \wedge B) \to (\neg A \vee \neg B))$
$T\neg(A \wedge B)$
$F\neg A \vee \neg B$
$FA \wedge B$
$F\neg A$
$F\neg B$
TA
TB

FA	FB
$\times$	$\times$

$$\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A, \neg A}}{\vdash A, (\neg A \vee \neg B)} \qquad \dfrac{\dfrac{B \vdash B}{\vdash B, \neg B}}{\vdash B, (\neg A \vee \neg B)}}{\dfrac{\dfrac{\vdash A \wedge B, (\neg A \vee \neg B)}{\neg(A \wedge B) \vdash (\neg A \vee \neg B)}}{\vdash \neg(A \wedge B) \to (\neg A \vee \neg B)}}$$

5. The cut rule corresponds to growth of deduction at the root:

$$\dfrac{\underline{A} \vdash A \qquad B \vdash C}{\underline{A}, A \to B \vdash C}\,\mathrm{L}{\to}$$

forms a deduction

$$\begin{array}{c} \underline{A} \\ \vdots \\ \dfrac{A \qquad A \to B}{B}\,{\to}\mathrm{E} \\ \vdots \\ C \end{array}$$

growing towards the root

$$\dfrac{\dfrac{\underline{A} \vdash A \qquad B \vdash B}{\underline{A}, A \to B \vdash C}\,\mathrm{L}{\to} \qquad \underline{B} \vdash A \to B}{\underline{A}, \underline{B} \vdash B}\,\mathrm{cut}$$

corresponds to

$$\dfrac{\begin{array}{c}\underline{A} \\ \vdots \\ A\end{array} \qquad \begin{array}{c}\underline{B} \\ \vdots \\ A \to B\end{array}}{B}\,{\to}\mathrm{E}$$

Cut-elimination: To any proof of $\underline{A} \vdash \underline{B}$, there exists a cut-free proof$\underline{A} \vdash \underline{B}$.

We need to remove each

$$\frac{\underline{A} \vdash A^+, \underline{B} \qquad \underline{C}, A^+ \vdash \underline{D}}{\underline{A}, \underline{C} \vdash \underline{B}, \underline{D}}$$

Depening on A^+:

$A^+ = B \wedge C$:

$$\cfrac{\cfrac{\underline{A} \vdash B, \underline{B} \quad \underline{A'} \vdash C, \underline{B'}}{\underline{A}, \underline{A'} \vdash B \wedge C, \underline{B}, \underline{B'}}\ \mathrm{R}\wedge \qquad \cfrac{\underline{C}, B \vdash \underline{D}}{\underline{C}, B \wedge C \vdash \underline{D}}\ \mathrm{L1}\wedge}{\underline{A}, \underline{A'}, C \vdash \underline{B}, \underline{B'}, D}\ \mathrm{cut}$$

transforms into

$$\frac{\underline{A} \vdash B, \underline{B} \qquad \underline{C}, B \vdash \underline{D}}{\underline{A}, \underline{A'}, \underline{C} \vdash \underline{B}, \underline{B'}, \underline{D}}\ \mathrm{cut}$$

$A^+ = B \vee C$:

$$\cfrac{\cfrac{\underline{A} \vdash C, \underline{B}}{\underline{A} \vdash B \vee C, \underline{B}}\ \mathrm{R2}\vee \qquad \cfrac{\underline{C}, B \vdash \underline{D} \quad \underline{B'}, C \vdash \underline{D'}}{\underline{B}, \underline{B'}, B \vee C \vdash \underline{D}, \underline{D'}}\ \mathrm{L}\vee}{\underline{A}, \underline{A'}, C \vdash \underline{B}, \underline{B'}, D}\ \mathrm{cut}$$

transforms into

$$\frac{\underline{A} \vdash B, \underline{B} \qquad \underline{C}, B \vdash \underline{D}}{\underline{A}, \underline{A'}, \underline{C} \vdash \underline{B}, \underline{B'}, \underline{D}}\ \mathrm{cut}$$