Tutorial 1

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PROBABILITY THEORY – A CRASH COURSE

1.) PROBABILITY SPACE – a set of all possible outcomes of a random experiment

S

finite
countable → this course

uncountable

Example: a set of all n-bit strings

2.) Probability events: $E \subseteq S$

Example: a string with exactly 3 symbols '1'.

3.) Probability function $p: S \to [0,1]$

$$\forall i \in S \quad p(i) \geq 0$$

$$\sum_{i \in S} p(i) = 1$$

Example: $\boxed{p(x) = \frac{1}{2^n}}$ (uniform distribution over all strings x of size n.)

$$p(E) = \sum_{i \in E} p(i)$$

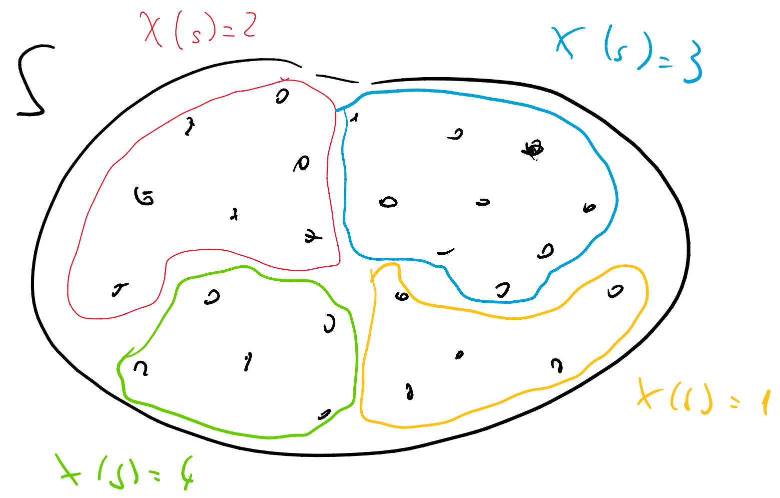
Example: what's the probability to obtain a 5-bit string with exactly 3 symbols '1'.

$$P(E) = \sum_{i \in E} p(i) = \sum_{i \in E} \frac{1}{32} = \binom{5}{3} \cdot \frac{1}{32} = \frac{10}{32}$$

$\boxed{\text{RANDOM VARIABLE}}$ (X, Y, Z)

$$X : S \to \mathbb{R}$$

Essentially X is a partition of S into mutually exclusive and collectively exhaustive set of events.



Example: X is a number of symbols '1' in an n-bit random string

Example: For $n=4$ what is the distribution of X?

$Pr(X=0) = 1/16$

$Pr(X=1) = 1/4$

$Pr(X>1) = 11/16$

$Pr(X=2) = 6/16$ $\qquad$ $Pr(X \geq 2 \wedge X < 4) = 10/16$

$Pr(X=3) = 1/4$

$Pr(X=4) = 1/16$

Expectation of r.v.s

$$\mu(X) = E(X) = \sum_{\substack{i \in \mathbb{R} \\ i \in Im(X)}} i \cdot Pr(X=i)$$

__Example!__ $E(X) = \sum_{i=0}^{4} i \cdot Pr(X=i) = 0 \cdot 1/16 + 1 \cdot \frac{4}{16} + 2 \cdot \frac{6}{16} + 3 \cdot \frac{4}{16} + 4 \cdot \frac{1}{16}$

$$= 2$$

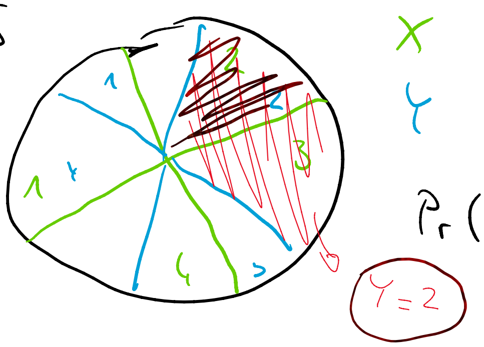
Conditional probabilities

Given 2 random variables X and Y over the same random experiment

define conditional probability of X given $Y=j$

$$Pr(X=i \mid Y=j) = Pr(X=i \wedge Y=j) \,/\, Pr(Y=j) \qquad [Pr(Y=j) \neq 0]$$

S



$Pr(X=2 \mid Y=2)$

Intuitively we are creating a new probability space equal to event

$Y = j$.

Example: $S = \{0,1\}^4$

$X =$ number of symbols '1'

$Y =$ parity of the string

$$\Pr(Y=0) = 1/2$$
$$\Pr(Y=1) = 1/2$$

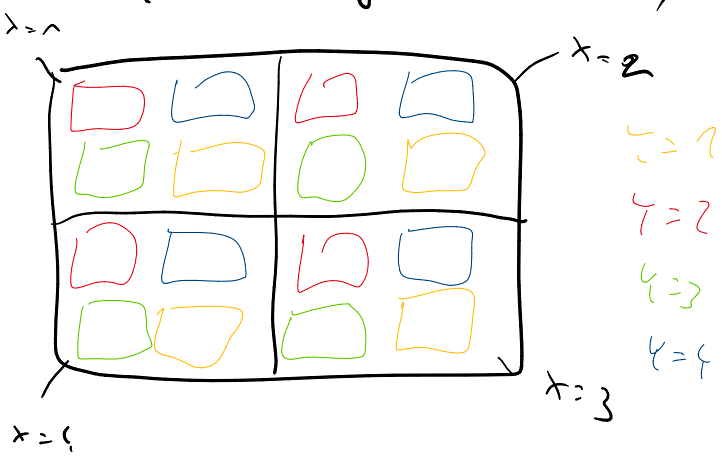
$$\Pr[X=3 \mid Y=0] = 0 = \Pr\{X=3 \wedge Y=0\}$$

$$\Pr[X=3 \mid Y=1] = \frac{\Pr[X=3, Y=1]}{\Pr[Y=1]} = \frac{4/16}{1/2} = \frac{8}{16} = 1/2$$

$\boxed{\text{Independence of random variables}}$

X and Y are independent if for all $i, j \in \mathbb{R}$

$$\Pr(X=i \mid Y=j) = \Pr(X=i)$$



Example: Are X and Y from the previous example independent?

$$\Pr(X=3) = 1/4 \neq \Pr(X=3 \mid Y=0) = 0$$

Z - the value of the first bit of $\{0,1\}^4$

Are Z and Y independent?

$Pr\{Z=1\} = 1/2$

$Pr\{Z=0\} = 1/2$

$Pr\{Z=1 \mid Y=1\} = 1/2$

$Pr\{Z=0 \mid Y=1\} = 1/2$

$Pr[Z=1 \mid Y=0\} = 1/2$

$Pr\{Z=0 \mid Y=0\} = 1/2$

1000	0111
1110	0100
1101	0010
1011	0001

1111
1001
1010
1100

LINEARITY OF EXPECTATION

$$W = X + Y + Z$$

$$E(W) = E(X+Y+Z) = E(X) + E(Y) + E(Z)$$

Example

$W = X + Y + Z$

$$E(W) = \sum_{i,j,k} (i+j+k) \cdot Pr(X=i, Y=j, X=k)$$

$$= \underset{\substack{\| \\ 2}}{E(X)} + \underset{\substack{\| \\ 1/2}}{E(Y)} + \underset{\substack{\| \\ 1/2}}{E(Z)} = 3$$

$$E(X_1 \cdot X_2) \neq E(X_1) E(X_2)$$

The law of total probability

r.v. X and Y

$$\Pr(X=i) = \sum_{j \in \mathrm{Im}(Y)} \Pr(X=i \mid Y=j) \cdot \Pr(Y=j)$$

$$= \sum_{j \in \mathrm{Im}(Y)} \Pr(X=i \wedge Y=j)$$

RQUICKSORT

IN: A collection of numbers S

OUT: Ordered list of numbers in S

1.) if S contains a single element output S

2.) choose a pivot $z \in S$ uniformly at random

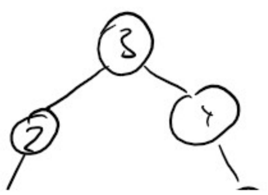
3.) Create S_1 which contain all $s \in S$ $s < z$

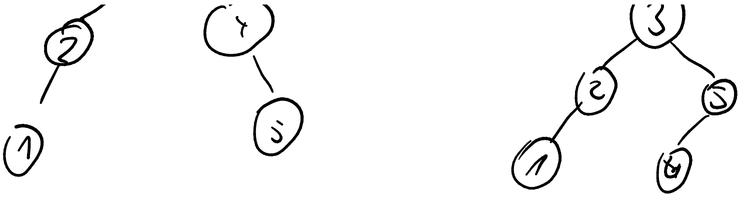
Create S_2 which contains all $s \in S$ $s > z$

4.) Output (rquicksort (S_1), z, rquicksort (S_2))

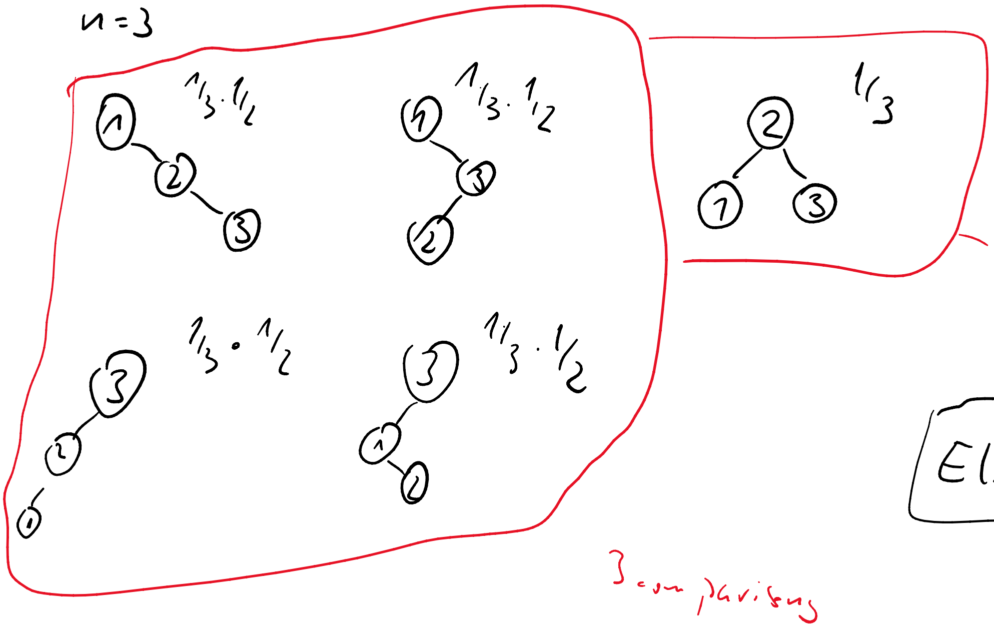
$S = \{1, 2, 3, 4, 5\}$

each run can be identified with an ordered binary tree





$n=3$



2 comparisons

$$\boxed{E(X) = 4 \cdot \tfrac{1}{6} \cdot 3 + \tfrac{1}{3} \cdot 2 = \frac{8}{3}}$$

X_i: number of comparisons with input of size i.

$$f(i) \geq E(X_i)$$

Can we find an upper bound $f(i)$ on $E(X_i)$ that shows $E(X_i)$ scales well?

$$f(i) \in O(i \log i)$$