Tutorial 6

29 March 2023 14:08

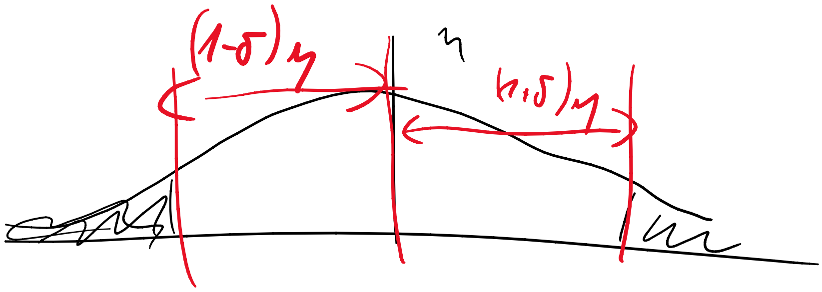
Concentration bounds

-> Chernoff bound applications

  -> Algorithm to estimate π

-> Introduction to routing on hypercubes

Chernoff bound



$$X = \sum_{i=1}^{n} X_i$$

X_i - Bernoulli r.v. (take value 0 or 1)

w.p. $Pr(X_i = 1)$ p

$E(X_i) = p$

$E(X) = n.p = \mu$

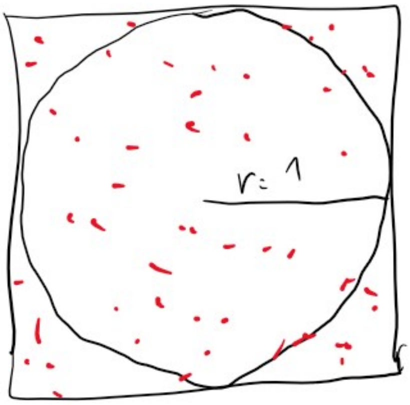
$$Pr(X \le (1-\delta)\mu) \le e^{-\frac{\mu\delta^2}{2}} \le e^{-\frac{\mu\delta^2}{3}}$$

$$Pr(X \ge (1+\delta)\mu) \le e^{-\frac{\mu\delta^2}{3}}$$

$$Pr(|X-\mu| \ge \delta\mu) \le 2 \cdot e^{-\frac{\mu\delta^2}{3}}$$

$$0 \le \delta \le 1$$

Algorithm to estimate π



$z_i = 1$ if i^{th} point is inside the circle

$z_i = 0$ otherwise

$$Pr(z_i = 1) = \frac{A(\text{circle})}{A(\text{square})} = \frac{\pi r^2}{4} = \frac{\pi}{4}$$

After n samples $z = \sum_{i=1}^{n} z_i$

$$\underline{\underline{E(z) = \frac{n \cdot \pi}{4}}}$$

$$z' = \frac{4 \cdot z}{n} \qquad E(z') = \pi$$

$$Pr\left(|z' - \pi| \geq \varepsilon \cdot \pi \right)$$

$$\shortparallel$$

$$Pr\left(\left| \frac{n}{4} z' - \frac{n}{4} \pi \right| \geq \frac{n}{4} \varepsilon \cdot \pi \right)$$

$$\shortparallel$$

$$|z - E(z)| \geq \varepsilon E(z) \leq 2 \cdot e^{-\frac{\frac{n}{4}\pi \varepsilon^2}{3}} = 2e^{-\frac{n\pi\varepsilon^2}{12}}$$

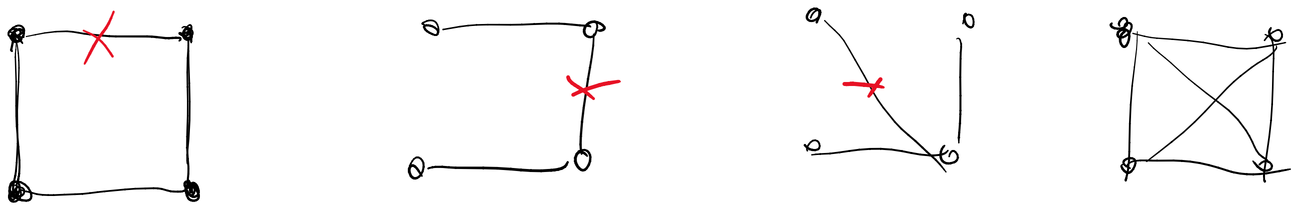
$$\shortparallel$$

$$Pr(z_i = 1) = \frac{V(\text{sphere})}{V(\text{cube})} = \frac{4/3 \, \pi r^3}{8} = \frac{\pi}{6}$$

$$\leq 2 e^{-\frac{\frac{n}{6}\pi \varepsilon^2}{3}} = 2 e^{-\frac{n\pi\varepsilon^2}{18}}$$

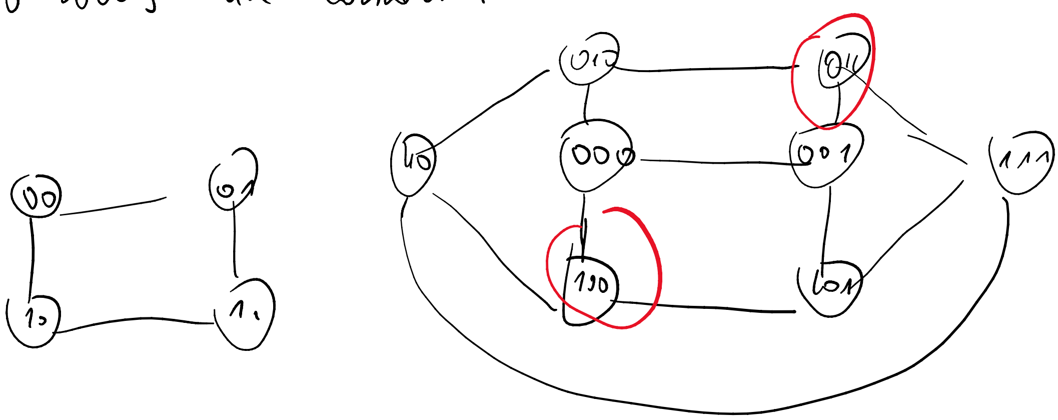
$$\leq 2e^{-\frac{\frac{n}{6}\pi\varepsilon^2}{3}} = 2e^{-\frac{n\pi\varepsilon^2}{18}}$$

Routing on hypercubes (Introduction)



$N = 2^d$ nodes labeled by binary strings of length d

Two nodes are connected iff their labels differ in exactly 1 bit



Total number of edges is $\frac{N \cdot d}{2} \sim O(n \log N)$,

Packet$(i, X, d(i))$

i: source

$d(i)$: destination

Assumption: In each step only 1 packet can be sent through a line

Deterministic routing: left to right bit-fixing

LTR

$1100 \to 0100 \to 0101$

$\searrow 1101 \nearrow$

($\ell!$ different routes of length ℓ where ℓ is the Hamming distance)

Experiment to test throughput of routing algorithms

Each node gets a packet with a destination.

How long will it take to deliver all the packets.

$\boxed{\text{The bad case}}$

$x_1, \dots, x_d \to x_1, \dots, x_1 \Rightarrow$ large subset of packets need to pass through $000\dots0$

$i = 1100 \to d(i) = 0011$

$j = 0100 \to d(j) = 0010$

$\ell = 1000 \to d(\ell) = 0001$

L→R

$P_i : 1100 \to 0100 \to 0000 \to 0010 \to 0011$

$P_j : 0100 \to 0000 \to 0010$

$P_\ell : 1000 \to 0000 \to 0001$

$$\underbrace{x_1 \dots x_{d/2}}_{d/2}\ \underbrace{00 \dots 0}_{d/2} \to \underbrace{000}_{d/2}\ \underbrace{x_{d/2} \dots x_1}_{d/2}$$

All $2^{d/2}$ messagess pass trough $0000\dots0$ node and it can send at most d packets simultaneously $\Omega\left(\frac{2d}{d}\right)$ steps are needed.

$\sigma(i) \to$ intermediate destination chosen at random

$$i \xrightarrow{LTR} \sigma(i) \xrightarrow{LTR} d(i)$$

$\boxed{\text{w.p. at least } 1-2^{-5d} \text{ every packet gets delivered in time } 14d.}$