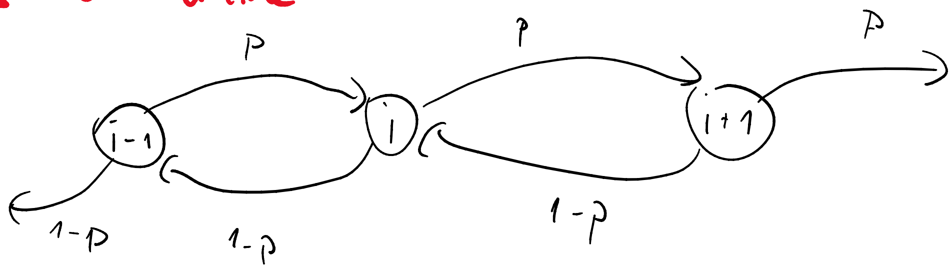
Tutorial 9

03 May 2023 14:09

MARKOV CHAINS

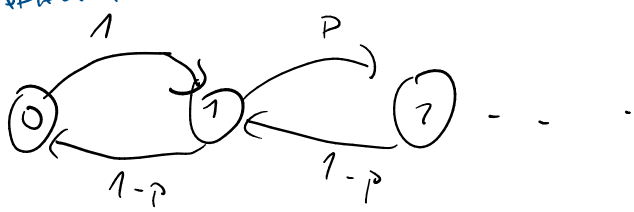
- → Walks on a line
- → Randomized algorithm for 2-SAT (3-SAT)
- → Fair 2-colorability of 3-colorable graphs ←

Walks on a line

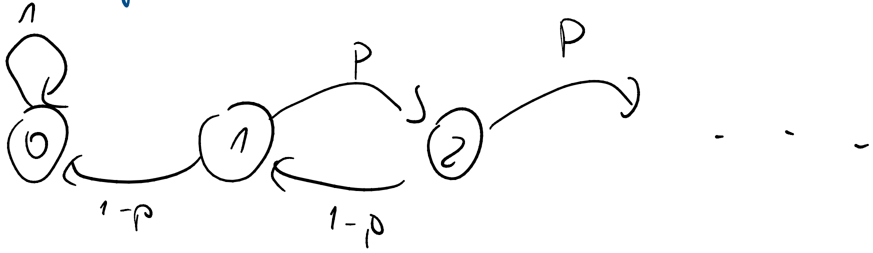


if the line is not infinite in both directions, it contains at least one barrier.

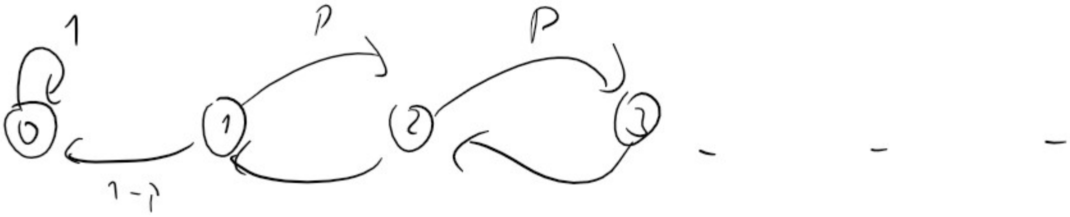
Reflecting



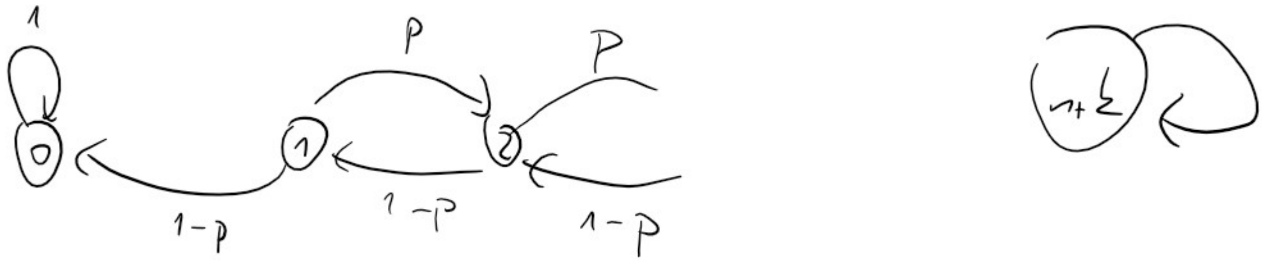
Absorbing



Monkey on a cliff



Gamler's ruin



TODAY

▷ What is the expected time to get from 0 to n.

$E_{i,j}$ = expected number of steps to get from i to j.

$E_{j,j} = 0$

$E_{0,1} = 1$

$$E_{0,n} = E_{0,1} + E_{12} + E_{23} + \dots + E_{n-1,n}$$

$$\forall_i \; E_{i,i+2} = E_{i,i+1} + E_{i+1,i+2}$$

$$\forall_i \; E_{i,i+1} = 1 + p\underbrace{E_{i+1,i+1}}_{=0} + (1-p)\,E_{i-1,i+1}$$

$$E_{i,i+1} = 1 + (1-p)\left(E_{i-1,i} + E_{i,i+1}\right)$$

$$p E_{i,i+1} = 1 + (1-p) E_{i-1,i}$$

$$E_{i,i+1} = \frac{1}{p} + \frac{1-p}{p} E_{i-1,i}$$

$$\boxed{E_{i,i+1} = v_i}$$

$$\boxed{\begin{aligned} v_0 &= 1 \\ v_i &= 1/p + \frac{(1-p)}{p} v_{i-1} \end{aligned}}$$

Example: $p = 1/2$

$v_0 = 1$

$v_i = 2 + v_{i-1}$

$v_i = 2i + 1$

$v_0 = c_0$

$v_1 = c_1$

$v_2 = c_2$

$v_{i+3} = a_0 v_0 + a_1 v_1 + a_2 v_2 + c$

2-SAT

Logical formula with $x_1, \dots, x_n$

$$\underbrace{(x_1 \vee x_2)}_{\text{clause}} \wedge (x_3 \vee \neg x_2) \wedge \dots \wedge (\neg x_1 \vee \neg x_5)$$

Is this formula satisfiable? → is there a truth assignment (0/1) to variables x_i such that the clauses are satisfied?

Randomized procedure

1.) Assign truth values at random

2.) Find an unsatisfied clause (if there is no such a clause we have a solution) with variables x_a and x_b.
Choose at random x_a or x_b and flip its truth value.

· If the "correct" assignment A exists, how long will it take to find it?

MC counts the number of variables in the intermediate assignment identical to A.

$(x_a \vee x_b)$ → if this is not satisfied, then <u>AT LEAST</u> one of x_a or x_b has a value different from A.

The upper bound on $E_{0,n}$ is the upper bound on the number of repetitions

$E_{i,i+1} = 2i+1$

$E_{0,1} = 1$

$$E_{0,n} = \sum_{i=0}^{n-1} E_{i,i+1} = \sum_{i=0}^{n-1} 2i+1 = n + 2\sum_{i=0}^{n-1} i$$

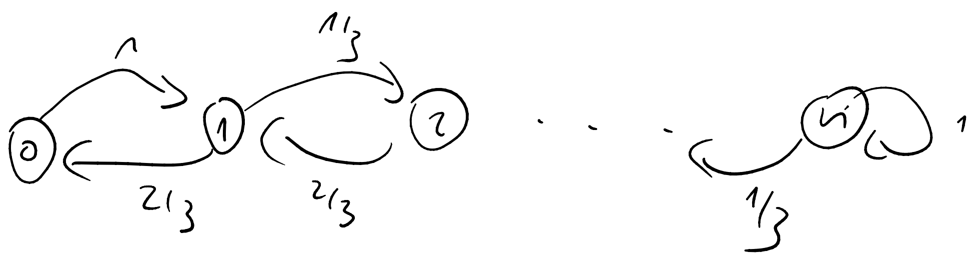
$$= n + \cancel{2}\,\frac{n(n-1)}{2} = n^2$$

The complete algorithm

Run the procedure $2n^2$ times. if the solution is not found say "I don't know"

Why does this fail for 3-SAT?

$(x_a \vee x_b \vee x_c)$ clause now has 3 variables



$E_{0,1} = 1$

$$E_{i,i+1} = 1/p + \frac{1-p}{p} E_{i-1,i}$$

for $p = 1/3$

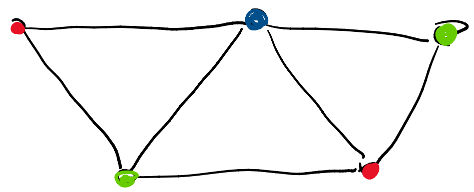
$r_0 = 1$

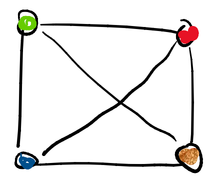
$r_i = 3 + 2r_{i-1}$

$r_i = 2^{i+2} - 3$

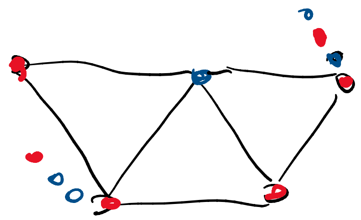
$\Rightarrow$ the number of

Let G be a 3-colorable graph





Task: Find a 2-coloring of G, such that there is no monochromatic triangle. (fair 2-coloring)

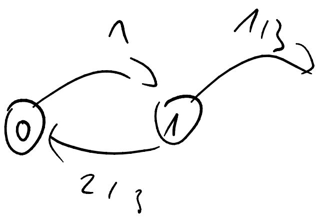


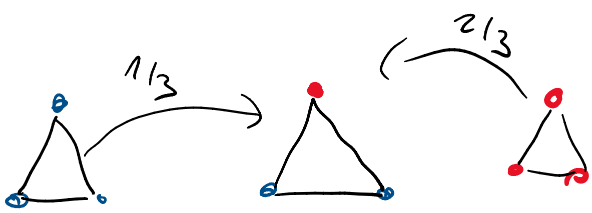
it exists: 3-coloring to 2-fair-coloring :-> choose 1 of the colors an change it to one of the others.

Randomized procedure

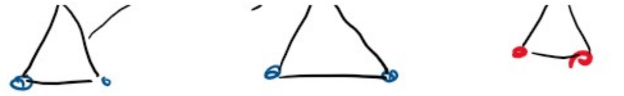
1.) Choose a random 2-coloring

2.) find a monochromatic triangle, choose one of it's vertices at random and flip it's color.



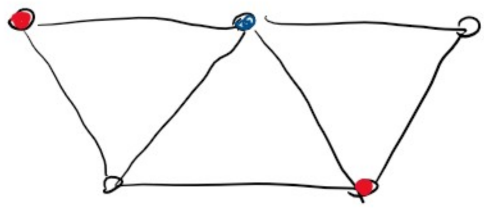


← wrong, leads to 3-SAT solution



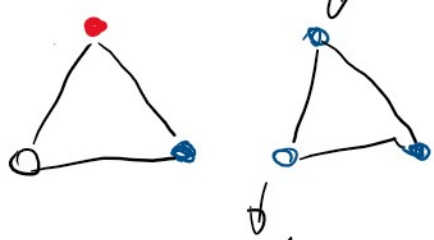
3-SAT solution

What is important is not the correct 2-coloring itself, but the partial 2-coloring one obtains from 3-coloring



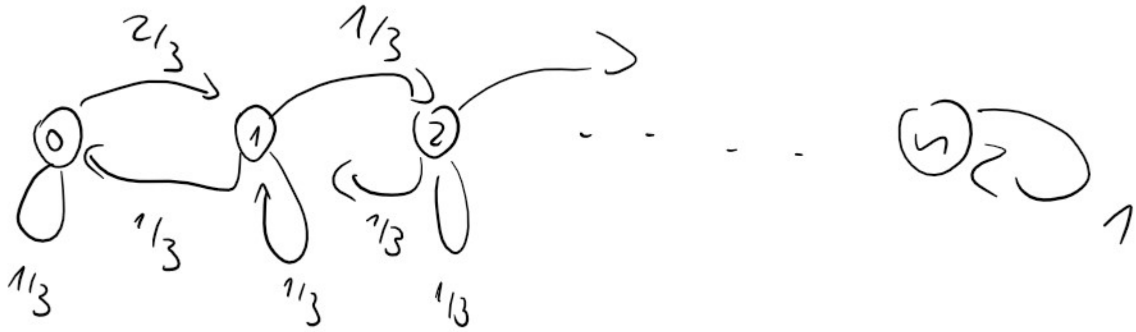
I don't care about this color

i get closer



i get farther

does not change the distance!



$E_{0,1} = 3/2$

$E_{i,i+1} = 3 + E_{i-1,i}$

$V_n = 3n$

$v_0 = 3/2$

$v_i = 3 + v_{i-1}$

$v_i = 3i + 3/2$

$$E_{0,n} = \sum_{i=0}^{n-1} E_{i,i+1} = \frac{3n}{2} + 3 \sum_{i=0}^{n-1} i \in O(n^2)$$