Network Controllability

IV124

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Studying complex networks

1. Understand & Describe (Quantify)
2. Predict
3. Control

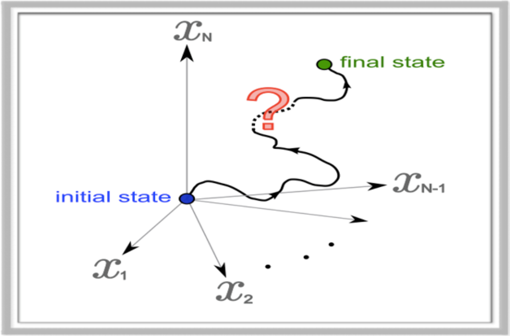
Research Questions

- Which nodes to target?
- How many nodes do we need to control the network?
- Are some networks easier to control than others?

Controllability

Controllability

A system is controllable if it can be driven from any initial state to any desired final state.



Controllability

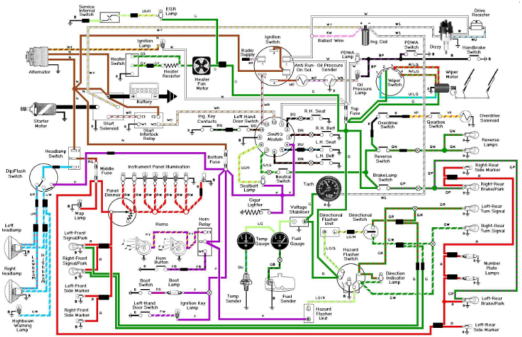
Even a simple oldtimer car can have several thousands of components.

Yet you only need to manipulate three components to control the car (gas, brake, steering wheel).

Controllability

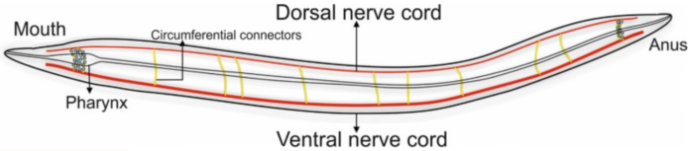
With real-world networks, we are facing a inverse problem:

- We have a network, but we do not know which components control the system
- We need to identify the driver nodes (N_D)



Case study

Control of Neuronal Network in C. elegans[1]



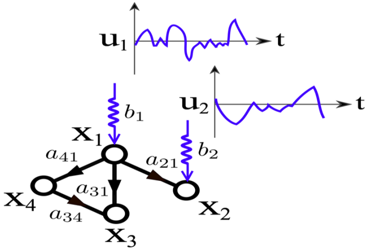
- body $\approx$ 1000 cells
- brain: N $\approx$ 300; E $\approx$ 2500
- scale-free structure
- 16 % driver nodes ($\approx$ 50 neurons)
- driver nodes avoid hubs

[1] Badhwar R, Bagler G. 2015. Control of Neuronal Network in Caenorhabditis elegans.

Controlling simple linear system

$$\frac{dX}{dt} = A \times X(t) + B \times u(t)$$

- $A \in R^{N \times N}$: adjacency matrix
- $X(t) \in R^{N \times 1}$: state vector
- $u(t) \in R^{M \times 1}$: input vector (signal)
- $B \in R^{N \times M}$: input matrix



$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ a_{31} & 0 & 0 & a_{34} \\ a_{41} & 0 & 0 & 0 \end{pmatrix}; \ \mathbf{B} = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Note: Oriented network, only incoming links count

Transpose of the weighted adjacency matrix.

Kalman's Rank Condition[2]

Kalman's Rank Condition:

A system is controllable if its controllability matrix has full rank.

rank $C = N$
$C = [B, A \times B, A^2 \times B, \cdots, A^{N-1} \times B]$

Informally, every row (column) is linearly independent of one another.

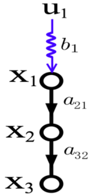
[2] Kalman, R.E. 1963. Mathematical description of linear dynamical systems

Example 1: Controllable

Consider a single - input discrete - time system : $N = 3$

$$\mathbf{X}(t+1) = \mathbf{A}\cdot\mathbf{X}(t) + \mathbf{b}\,u(t)$$

with $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}$.



Assume $\mathbf{X}(t=0) = \mathbf{0}$. Then we have the state sequence

$$\mathbf{X}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{X}(1) = \begin{bmatrix} b\,u(0) \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{X}(2) = \begin{bmatrix} b\,u(1) \\ a_{21} b\,u(0) \\ 0 \end{bmatrix}, \quad \mathbf{X}(3) = \begin{bmatrix} b\,u(2) \\ a_{21} b\,u(1) \\ a_{32} a_{21} b\,u(0) \end{bmatrix} = \begin{bmatrix} b & 0 & 0 \\ 0 & ba_{21} & 0 \\ 0 & 0 & ba_{32}a_{21} \end{bmatrix} \cdot \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} \cdots$$

Controllability Matrix:

$$\mathbf{C} = [\mathbf{B},\ \mathbf{A}\cdot\mathbf{B},\ \mathbf{A}^2\cdot\mathbf{B}, \cdots, \mathbf{A}^{N-1}\cdot\mathbf{B}]$$

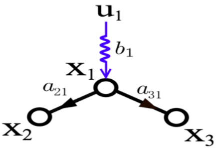
$$\mathbf{C} = [\mathbf{b},\ \mathbf{A}\cdot\mathbf{b},\ \mathbf{A}^2\cdot\mathbf{b}]$$

Example 2: Uncontrollable

Consider a single - input discrete - time system

$$\mathbf{X}(t+1) = \mathbf{A}\cdot \mathbf{X}(t) + \mathbf{b}\,u(t)$$

with $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}$.



Assume $\mathbf{X}(t=0) = \mathbf{0}$. Then we have the state sequence

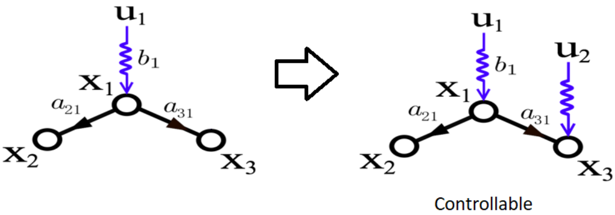
$$\underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{X}(0)} \quad \underbrace{\begin{bmatrix} b\,u(0) \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{X}(1)} \quad \underbrace{\begin{bmatrix} b\,u(1) \\ a_{21} b\,u(0) \\ a_{31} b u(0) \end{bmatrix}}_{\mathbf{X}(2)} \quad \underbrace{\begin{bmatrix} b\,u(2) \\ a_{21} b\,u(1) \\ a_{31} b\,u(1) \end{bmatrix}}_{\mathbf{X}(3)} = \begin{bmatrix} b & 0 & 0 \\ 0 & ba_{21} & 0 \\ 0 & ba_{31} & 0 \end{bmatrix} \cdot \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} \quad \cdots$$

Controllability Matrix:

$$\mathbf{C} = [\mathbf{b}, \mathbf{A}\cdot\mathbf{b}, \mathbf{A}^2\cdot\mathbf{b}]$$

Example 2: How to control it?

We cannot change the topology, so we need to send a signal to an additional node.



Driver Nodes – N_D

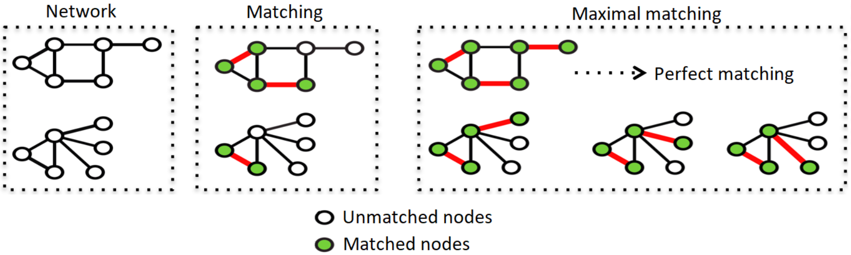
- What's the minimum number of N_D?
- How to efficiently identify them?
- Which network characteristics determine N_D?

Challenges with identification of N_D

- Link weights of real-world networks are usually unknown
- Brute-force search is not feasible, as there are $2^N - 1$ combinations
- Kalman's rank condition is hard to check for large systems, as it has a dimension of $N \times NM$

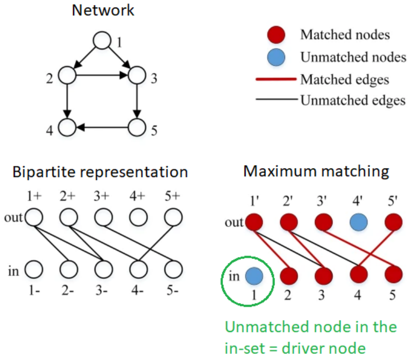
Solution: Graph Matching theory

- Matching $M \subseteq E$ is a set of links that don't have common nodes
- Maximal matching – a matching with the highest link count (more than one can be identified)
- Perfect matching – a matching that covers all nodes (there are no unmatched nodes)



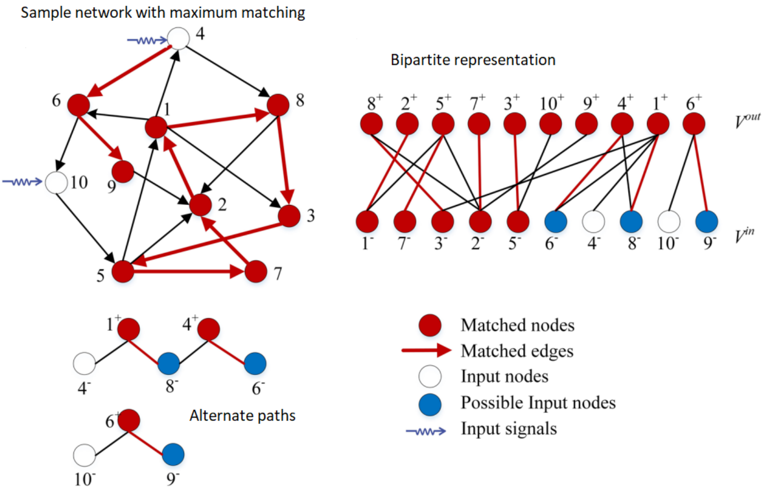
Matching in Directed Network[3]

The bipartite graph is built by splitting the node set N into two node sets N^{in} and N^{out}



[3]Zhang, Han & Zhang. 2015. An efficient algorithm for finding all possible input nodes for controlling complex networks

Matching in Directed Network[4]

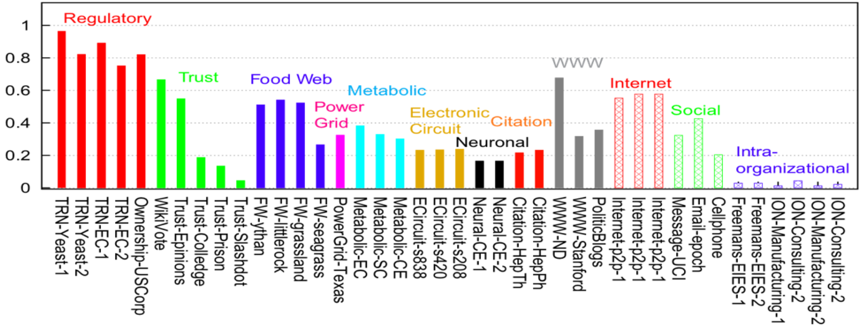


[4] Zhang, Han & Zhang. 2015. An efficient algorithm for finding all possible input nodes for controlling complex networks

Time Complexity Issue

- Brute force $\mathcal{O}(2^N)$
  - $\approx 10^{30}$ for $N = 100$
  - not feasible
- Hopcroft-Karp Algorithm
  - $\mathcal{O}(\sqrt{N}L)$ in worst case
  - $\mathcal{O}(logNL)$ in sparse graphs
  - Fast enough even for $N \approx 10^6$

N_D in real networks



- there is no observable trend across different networks
- regulatory networks have high $N_D \approx 0.8$
- social networks display lowest N_D

Hub controversy

Are hubs N_D or not?

Liu, Slotine, Barabási. Controllability of complex networks. 2011

- N_D tend to avoid hubs
- amount of N_D depends on degree distribution
- sparse and heterogeneous networks are harder to control than dense and homogeneous

Cowan et al. Nodal Dynamics, Not Degree Distributions, Determine the Structural Controllability of Complex Networks. 2012

- Signal to power dominating set is enough to control most complex networks

Control Centrality Measure[5]

Control Centrality

Control centrality of node *i* captures the controllable subspace's dimension or the controllable subsystem's size when we control node *i* only.

Reminder:
System dynamics: $\frac{dX}{dt} = A \times X(t) + B \times u(t)$
Kalman Controllability Matrix:
$rankC = [B, A \times B, A^2 \times B, \cdots, A^{N-1} \times B]$

When we control node *i* only, B reduces to single non-zero value vector $b^{(i)}$, and C becomes $C^{(i)}$

[5] Liu, Slotine, Barabási. 2012. Control Centrality and Hierarchical Structure in Complex Networks

Control Centrality Measure

rank$C^{(i)}$ can be used as a measure indicating the ability of the node to control the system.

- $C^{(i)} = N$ means that node i may control the whole system
- $C^{(i)} < N$ indicates a fraction of network that node i may control

Hence, control centrality measure C_c may be defined as
$C_c \equiv rank_g(C^{(i)})$

Control Centrality: Application[6]

A targeted attack on a malicious network aiming to damage their controllability.

Challenge: To target an attack, we need to know the network's adjacency matrix, which is often not known in real systems.

Solution: **Random upstream attack**

- Randomly choose a fraction of nodes P
- For each of chosen nodes, remove one of the incoming (upstream) neighbors
- If there are no incoming neighbors, remove the chosen node

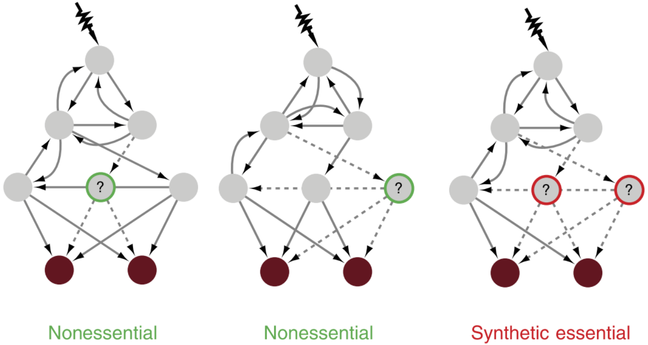
A random upstream attack is almost as good as a targeted attack (C_c)

[6] Liu, Slotine, Barabási. 2012. Control Centrality and Hierarchical Structure in Complex Networks

Synthetic Ablation[7]

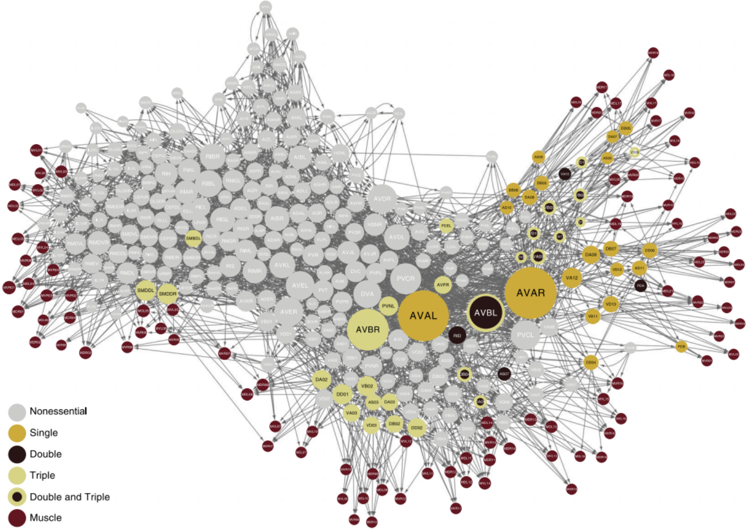
Synthetic lethality

Simultaneous knockout of two otherwise nonessential genes (or neurons) is lethal to the organism.



[7] Towson, Barabási. 2020. Synthetic ablations in the C. elegans nervous system

Synthetic Ablation



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