Vektorové prostory

• $\mathbb{R}^n$ $\quad (a_1,\dots,a_n)+(b_1,b_2,\dots,b_n)=(a_1+b_1,a_2+b_2,\dots,a_n+b_n)$

$c\in\mathbb{R}$: $\quad c\cdot(a_1,\dots,a_n)=(c\cdot a_1,\dots,c\cdot a_n)$

$$c\cdot(v+u)=c\cdot v+c\cdot u$$

• $\mathbb{R}[x]$ $\quad (a_0+a_1x+a_2x^2+\dots+a_nx^n)+(b_0+b_1x+\dots+b_nx^n)=a_0+b_0+(a_1+b_1)\cdot x+\dots+(a_n+b_n)\cdot x^n$

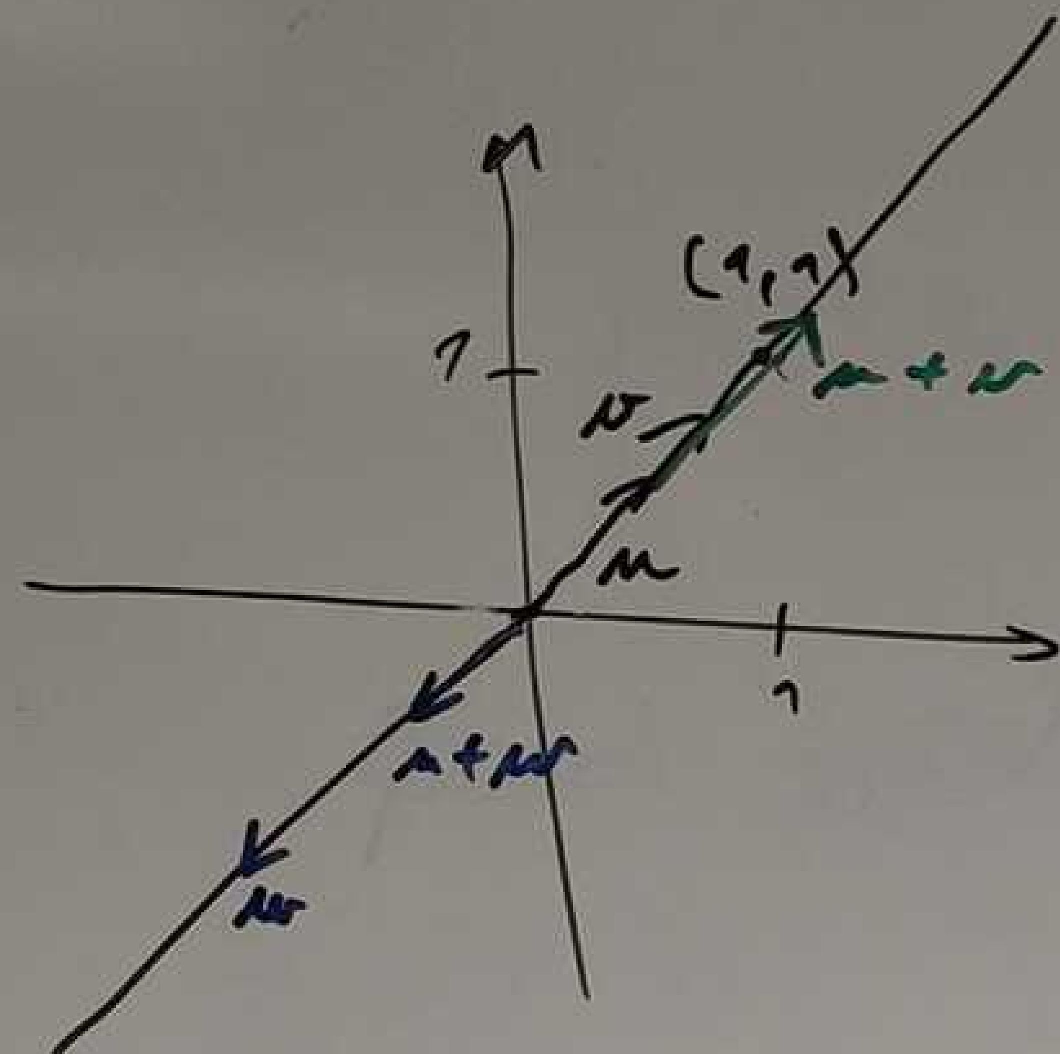
$c\in\mathbb{R}$: $\quad c\cdot(a_0+a_1x+\dots+a_nx^n)=c\cdot a_0+ca_1x+\dots+ca_nx^n$

• $\mathrm{Mat}_{3\times3}(\mathbb{R})$:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} & \dots \\ \vdots & \ddots & \\ & & \ddots \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots \\ & \ddots & \end{pmatrix}$$

$$c\cdot\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}=\begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & & \\ & \ddots & \end{pmatrix}$$

Vekt. podpr. v $\mathbb{R}^2$



$2\cdot(1,1)=(2,2)$

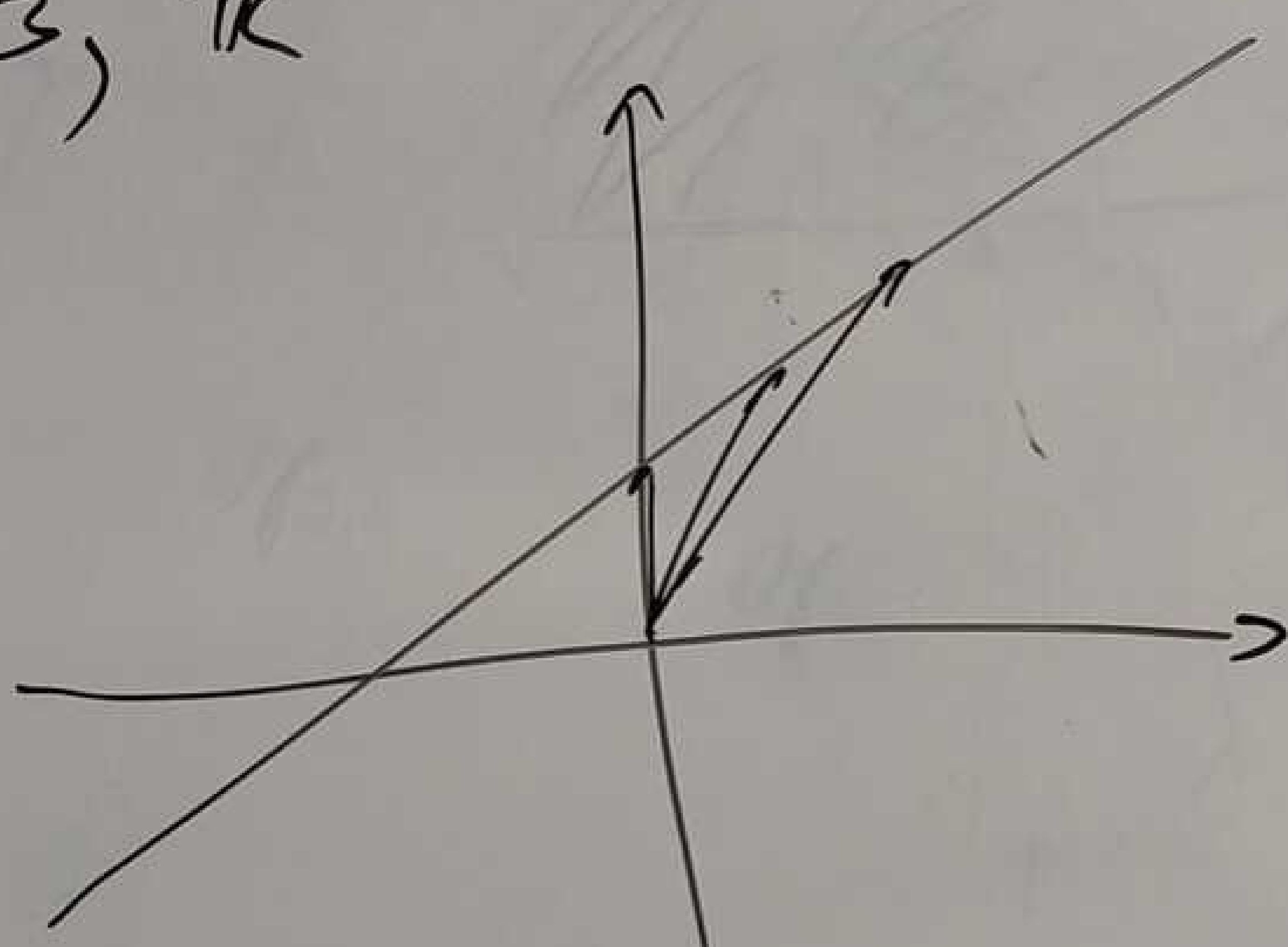
$t\cdot(1,1)=(t,t)$

$t\cdot(a,b)=(ta,tb)$

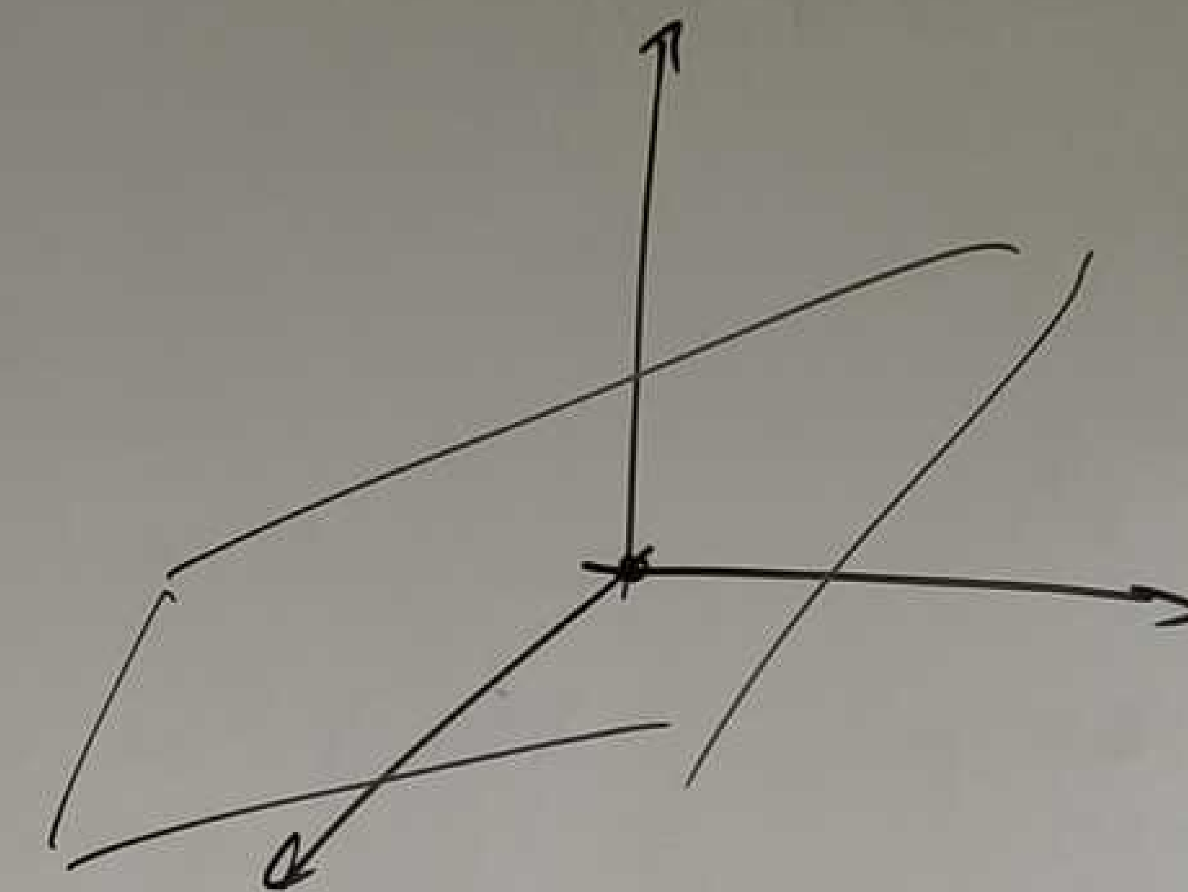
1) $\{(0,0)\}$ $\quad c\cdot(0,0)=(c\cdot 0,c\cdot 0)=(0,0)$

2) $\{(a\cdot t, b\cdot t) \mid t\in\mathbb{R}\}$

3) $\mathbb{R}^2$



Vekt. podp. v $\mathbb{R}^3$



$ax+by+cz=0$

$\frac{ax+by}{-c}=z$

1) $\{(0,0,0)\}$

2) $\{(t\cdot a, t\cdot b, t\cdot c) \mid t\in\mathbb{R}\}$

3) $\{(t, s, \frac{a\cdot t+b\cdot s}{-c}) \mid t,s\in\mathbb{R}\}$

4) $\mathbb{R}^3$

Je vektorovým podprostorem?

a) $U = \{ f \in \mathbb{R}[x] \mid f(3) = 0,\ f(-1) = 0 \} \subset \mathbb{R}[x]$

$(c \cdot f)(3) = 0$

$f, g \in U \ : \ f + g \overset{?}{\in} U \iff (f+g)(\underset{-1}{3}) = 0$ ✓

$(f+g)(\underset{-1}{3}) = f(\underset{-1}{3}) + g(\underset{-1}{3}) = 0 + 0 = \underset{0}{0}$

$(c \cdot f)(\underset{-1}{3}) = c \cdot f(\underset{-1}{3}) = 0$

<u>Např.</u>

$f = (x-3)^2 \cdot (x+1) \cdot x \in U$

$g = (x-3) \cdot (x+1) \cdot 2 \in U$

$f + g = (x-3)(x+1) \cdot (\ldots)$

b) $V = \{ A \in Mat_{2\times 2}(\mathbb{R}) \mid a_{11} + a_{22} = 1 \} \subset Mat_{2\times 2}(\mathbb{R})$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin V \Rightarrow$ není vekt. podpr.

$\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A \in V} + \underbrace{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}}_{B \in V} = \begin{pmatrix} a_{11}+b_{11} & \\ & a_{22}+b_{22} \end{pmatrix}$

$(a_{11}+b_{11}) + (a_{22}+b_{22})$

$= (a_{11}+a_{22}) + (b_{11}+b_{22}) = 2$

$\Rightarrow A + B \notin V$

$c \cdot \begin{pmatrix} a_{11} & \ldots \\ \ldots & a_{22} \end{pmatrix} = \begin{pmatrix} ca_{11} & \ldots \\ \ldots & ca_{22} \end{pmatrix}$

$c \cdot a_{11} + c \cdot a_{22} = c \cdot (a_{11} + a_{22})$

Př. Zjistěte, zda vektor $u = (1, -2, 3, 4) \in \mathbb{R}^4$

leží v lineárním obalu vektorů $\langle v_1, v_2, v_3 \rangle$

$v_1 = (1, 0, 1, -2)$

$v_2 = (3, -1, -1, -1)$

$v_3 = (0, 1, -5, 4)$

lin. obal: $a \cdot v_1 + b \cdot v_2 + c \cdot v_3 \quad , a, b, c \in \mathbb{R}$

(Chceme zjistit, zda $\exists a, b, c: \; a v_1 + b v_2 + c v_3 = u$

$$a \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix} + b \begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ -5 \\ 4 \end{pmatrix} \overset{?}{=} \begin{pmatrix} 1 \\ -2 \\ 3 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -2 \\ 1 & -1 & -5 & 3 \\ -2 & -1 & 4 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & -4 & -5 & 2 \\ 0 & 5 & 4 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -9 & 10 \\ 0 & 0 & 9 & -4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -9 & 10 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

soustava nemá řešení $\Rightarrow u \notin \langle v_1, v_2, v_3 \rangle$

$M \subset \mathbb{R}^5$

$M = \langle v_1, v_2, v_3, v_4 \rangle$

$v_1 = (1, 2, 1, 0, 1)$

$v_2 = (2, -1, 0, 1, 1)$

$v_3 = (1, -3, -1, 1, 0)$

$v_4 = (1, 7, 3, -1, 2)$

$v_1, \dots, v_4$ lin. nezávislé? (LN)

Jestli, vyberte bázi, a zbylé vektory vyjádřete v této bázi

LN $\Leftrightarrow$ $a v_1 + b v_2 + c v_3 + d v_4 = 0 \Rightarrow a = b = c = d = 0$

$$\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & -3 & 7 \\ 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -5 & -5 & 5 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix} \sim \begin{array}{cccc} a & b & c & d \\ v_1 & v_2 & v_3 & v_4 \end{array} \begin{pmatrix} \underline{1} & 2 & 1 & 1 \\ 0 & \underline{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$d = t$

$c = s$

$b = -c + d = -s + t$

$a = -2b - c - d$
$= s - 3t$

$\hookrightarrow v_1, v_2$ jsou LN

$M = \langle v_1, v_2 \rangle$

$v_3 = A v_1 + B v_2$

$v_4 = C v_1 + D v_2$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & -1 & -3 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right) \sim \begin{array}{c} A \quad B \\ \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \end{array}$$

$B = 1$

$A = 1 - 2B$
$= -1$

$$\boxed{\begin{aligned} v_3 &= -v_1 + v_2 \\ v_4 &= 3 v_1 - v_2 \end{aligned}}$$

$-1 \quad 3 \quad s=0 \quad t=-1$

$a v_1 + b v_2 + c v_3 + d v_4 = 0$

$a = s - 3t = 0 - 3\cdot(-1) = 3$

$b = -s + t = 0 + (-1) = -1$

v_3 má souřadnice v bázi (v_1, v_2)
$(-1, 1)$

$-v_1 + 3 v_2 = v_4$

v_4 má souř. $(3, -1)$

Př. Spočtěte souřadnice polynomu $1+3x+5x^2+10x^3 =: p$

v bázi $\alpha = (1+x+2x^2-x^3,\ 1+2x+x^3,\ 1+x+3x^2-x^3,\ 2+2x+4x^2+5x^3)$

v prostoru $\mathbb{R}_3[x]$

$$a\cdot(1+x+2x^2-x^3)+b\cdot(1+2x+x^3)+c(1+x+3x^2-x^3)+d\cdot(2+2x+4x^2+5x^3)$$
$$= 1+3x+5x^2+10x^3$$

$$\begin{array}{c} x^0 \\ x \\ x^2 \\ x^3 \end{array}\left(\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 3 \\ 2 & 0 & 3 & 4 & 5 \\ -1 & 1 & -1 & 5 & 10 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & -2 & 1 & 0 & 3 \\ 0 & 2 & 0 & 7 & 11 \end{array}\right) \sim \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left(\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 7 & 7 \end{array}\right) \end{array}$$

$7d = 7$

$d = 1$

$c = 7$

$b = 2$

$a = 1 - b - c - 2d$

$= 1 - 2 - 7 - 2 = -10$

Souřadnice p v bázi α jsou $(-10, 2, 7, 1)$

Př. Najděte bázi řešení soustavy

$$2x_1 - 3x_2 + 4x_3 - 8x_4 + x_5 = 0$$
$$x_1 + 2x_2 - 3x_3 + x_4 + 5x_5 = 0$$

$$\begin{pmatrix} 2 & -3 & 4 & -8 & 1 \\ 1 & 2 & -3 & 1 & 5 \end{pmatrix} \sim \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & 2 & -3 & 1 & 5 \\ 0 & -7 & 10 & -10 & -9 \end{pmatrix} \end{array}$$

$x_5 = r$

$x_4 = s$

$x_3 = t$

$-7x_2 = -10x_3 + 10x_4 - 9x_5$

$x_2 = \frac{10}{7}t - \frac{10}{7}s + \frac{9}{7}r$

$x_1 = -2x_2 + 3x_3 - x_4 - 5x_5$

$= -\frac{20}{7}t + \frac{20}{7}s - \frac{18}{7}r$

$+3t - s - 5r$

$x_1 = \frac{1}{7}t + \frac{13}{7}s + \frac{17}{7}r$

$$x = (x_1, x_2, x_3, x_4, x_5) = \left(\frac{1}{7}t + \frac{13}{7}s + \frac{17}{7}r,\ \frac{10}{7}t - \frac{10}{7}s + \frac{9}{7}r,\ t, s, r\right)$$
$$= t\cdot\left(\frac{1}{7}, \frac{10}{7}, 1, 0, 0\right) + s\cdot\left(\frac{13}{7}, -\frac{10}{7}, 0, 1, 0\right) + r\cdot\left(\frac{17}{7}, \frac{9}{7}, 0, 0, 1\right)$$

Př: Bázi prostoru řešení

$$2x_1 - 3x_2 + 4x_3 - 8x_4 + x_5 = 0$$
$$x_1 + 2x_2 - 3x_3 + x_4 + 5x_5 = 0$$

$$\begin{pmatrix} 2 & -3 & 4 & -8 & 1 \\ 1 & 2 & -3 & 1 & 5 \end{pmatrix} \sim \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 1 & 2 & -3 & 1 & 5 \\ 0 & -7 & 10 & -10 & -9 \end{pmatrix} \end{array}$$

$x_5 = 7t$

$x_4 = 7s$

$x_3 = 7r$

$-7x_2 = -10x_3 + 10x_4 + 9x_5$

$x_2 = +10r - 10s - 9t$

$x_1 = -2x_2 + 3x_3 - x_4 - 5x_5$

$= -20r + 20s + 18t + 21r - 7s - 35t$

$= r + 13s - 17t$

$$x = (x_1, x_2, x_3, x_4, x_5) = (r + 13s - 17t,\ 10r - 10s - 9t,\ 7r,\ 7s,\ 7t)$$
$$= r\cdot(1, 10, 7, 0, 0) + s\cdot(13, -10, 0, 7, 0) + t\cdot(-17, -9, 0, 0, 7), \quad r, s, t \in \mathbb{R}$$