$\mathbb{R}^4$

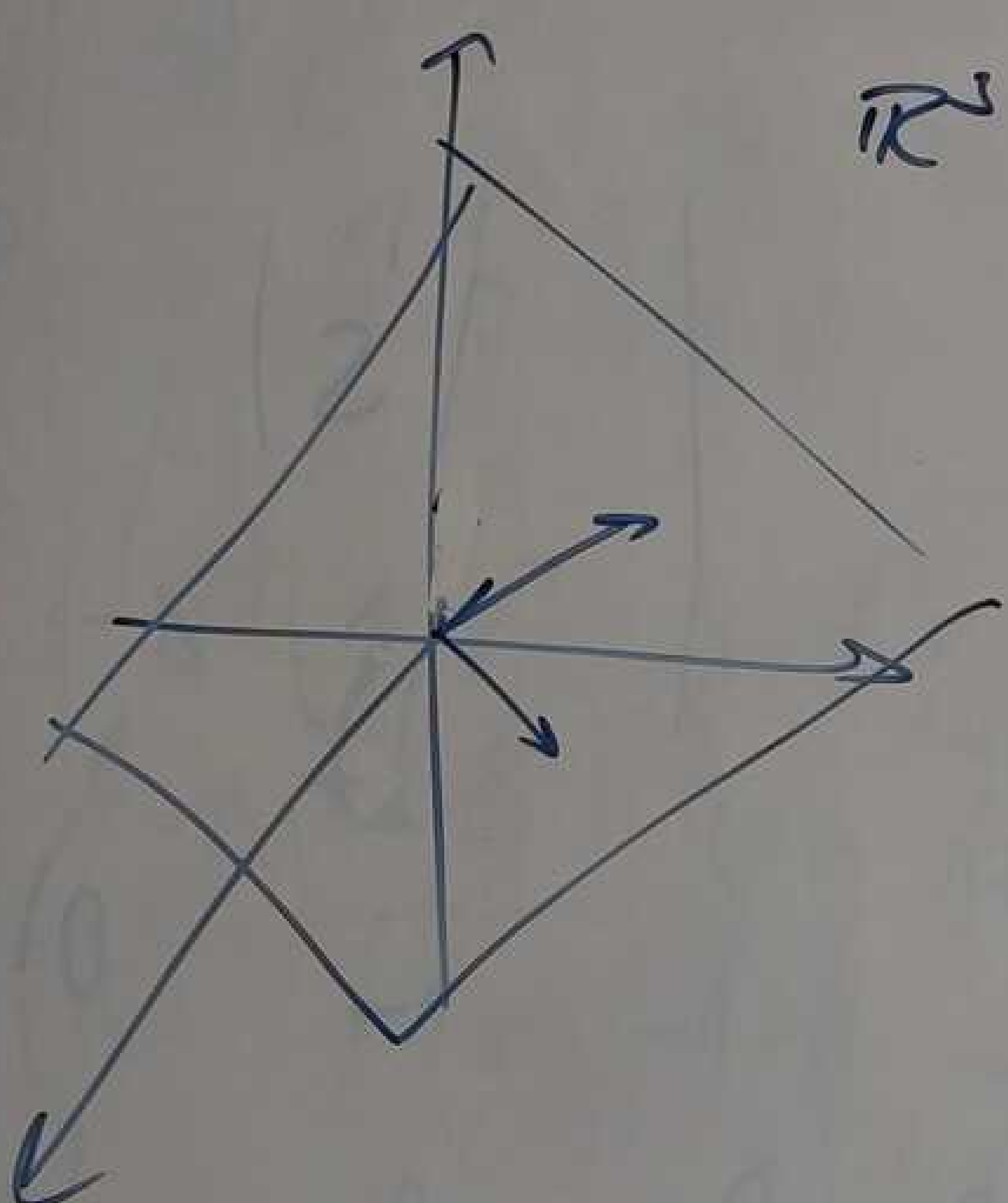
$$P = [\underset{u}{(4,0,-2,6)}, \underset{v}{(2,1,-2,3)}, \underset{w}{(3,1,-2,4)}]$$

$$Q = [\underset{u'}{(1,-1,0,2)}, \underset{v'}{(2,2,-1,3)}, \underset{w'}{(0,1,1,0)}]$$

$$P \cap Q = \{x \in \mathbb{R}^4 \mid x \in P \ \& \ x \in Q\}$$

$$x = \underline{au + bv + cw} = du' + ev' + fw'$$

$\mathbb{R}^3$



$$au + bv + cw - du' - ev' - fw' = 0$$

$$\begin{array}{cccccc} a & b & c & d & e & f \end{array}$$

$$\begin{pmatrix} 4 & 2 & 3 & -1 & -2 & 0 \\ 0 & 1 & 1 & 1 & -2 & -1 \\ -2 & -2 & -2 & 0 & 1 & -1 \\ 6 & 3 & 4 & -2 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & 2 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & -2 & -1 \\ 0 & -2 & -1 & -1 & 0 & -2 \\ 0 & -3 & -2 & -2 & 0 & -3 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & 2 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 & -4 & -4 \\ 0 & 0 & 1 & 1 & -6 & -6 \end{pmatrix}$$

$$\begin{array}{cccccc} a & b & c & d & e & f \end{array}$$

$$\sim \begin{pmatrix} 2 & 2 & 2 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 & -4 & -4 \\ 0 & 0 & 0 & 0 & -2 & -2 \end{pmatrix} \to -2e - 2f = 0$$

$$e = -f$$

$f = s$

$d = t$

$e = -s$

$\underline{c} = -d + 4e + 4f = \underline{-t}$

$\underline{b} = -c - d + 2e + f = t - t - 2s + s$

$\underline{= -s}$

$\underline{a} = \frac{1}{2}(-2b - 2c + e - f)$

$= \frac{1}{2}(2s + 2t - s - s) = \underline{t}$

$P \cap Q: \; tu - sv - tw = t(4,0,-2,6) - s(2,1,-2,3) - t(3,1,-2,4)$

$= t(1,-1,0,2) - s(2,1,-2,3)$

$P \cap Q = [(1,-1,0,2),(2,1,-2,3)]$ → má dimenzi 2

$P + Q =$ je generované u, v, w, u', v', w' ∈ není báze

Podle vedoucích koef. v upravené matici vlevo vidíme, že jsou LN u, v, w, v'

$P + Q = [u, v, w, v'] = \mathbb{R}^4$ → má dimenzi 4

$\varphi : U \to V$ lin. zobrazení

$$\varphi(\underset{\in U}{u} + \underset{\in U}{v}) = \underset{\in V}{\varphi(u)} + \underset{\in V}{\varphi(v)}$$

$$\varphi(0) = 0$$

$$\varphi(\underset{\in \mathbb{R}}{a} \cdot u) = a \cdot \varphi(u)$$

Př.

Určete, zda je lin. zobrazením:

1) $\varphi : \mathbb{R}^2 \to \mathbb{R}$ $\quad \varphi((x_1, x_2)) = \dfrac{x_1^3 + x_2^3}{x_1^2 + x_2^2}$ ✗ $\varphi(0)$ není def.

$$\varphi(1,2) + \varphi(1,3) = \frac{1+8}{1+4} + \frac{1+27}{1+9} = \frac{9}{5} + \frac{28}{10} = \frac{46}{10} = \frac{23}{5}$$

$$\varphi(2,5) = \frac{8+125}{4+25} = \frac{133}{29}$$

$\neq$

2) $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$ $\quad \varphi(x_1, x_2, x_3) = (2x_1 - x_2, 2x_2 - x_3)$ ✓

$$\varphi(a \cdot (x_1, x_2, x_3)) = \varphi(ax_1, ax_2, ax_3) = (2ax_1 - ax_2, 2ax_2 - ax_3) = a \cdot (2x_1 - x_2, 2x_2 - x_3)$$
$$= a \cdot \varphi(x_1, x_2, x_3)$$

$$\varphi((x_1, x_2, x_3) + (y_1, y_2, y_3)) = \varphi(x_1 + y_1, x_2 + y_2, x_3 + y_3) = (2(x_1 + y_1) - (x_2 + y_2), 2(x_2 + y_2) - (x_3 + y_3))$$
$$= ((2x_1 - x_2) + (2y_1 - y_2), (2x_2 - x_3) + (2y_2 - y_3)) = \varphi(x_1, x_2, x_3) + \varphi(y_1, y_2, y_3)$$

$$\varphi(x) = A x = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \varphi(x_1, x_2, x_3)$$

$$A = (\underset{\varphi\begin{pmatrix}1\\0\\0\end{pmatrix}}{\varphi(e_1)}, \underset{\varphi\begin{pmatrix}0\\1\\0\end{pmatrix}}{\varphi(e_2)}, \underset{\varphi\begin{pmatrix}0\\0\\1\end{pmatrix}}{\varphi(e_3)})$$

3) $\varphi: \mathbb{R}_3[x] \to \mathbb{R}^2 \quad \varphi(p) = (p(0), p'(0))$ ✓

$(p+q)' = p' + q'$

$(a \cdot p)' = a \cdot p', \quad a \in \mathbb{R}$

$A = (\varphi(x^3), \varphi(x^2), \varphi(x), \varphi(1))$

$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$\varphi(x^3 - x + 1) = (0^3 - 0 + 1, 3 \cdot 0^2 - 1) = (1, -1)$

$$\begin{array}{cccc} \varphi(x^3) & \varphi(x^2) & \varphi(x) & \varphi(1) \\ \end{array}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{array}{c} x^3 \\ x^2 \\ x \\ 1 \end{array} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Pr. Ve vekt. pr. $\mathbb{R}^3$ uvažme bázi $\alpha = (u_1, u_2, u_3)$

$u_1 = (1, -1, 1), \ u_2 = (1, 1, 0), \ u_3 = (2, 1, 1)$

Lin. zobr. $\varphi: \mathbb{R}^3 \to \mathbb{R}^3 \quad \varphi(u_1) = u_2, \ \varphi(u_2) = u_3, \ \varphi(u_3) = u_1$

Najděte matici A reprezentující zobr. ve standartní bázi

$\varphi(x) = Ax \qquad$ chcem $A = (\varphi)_{\varepsilon\varepsilon}$

$(\varphi)_{\alpha\alpha} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(Jednodušší postup o 2 strany dále)

$$\underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}}_{(\varphi)_{\varepsilon\alpha}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

souř. b_α $\qquad$ st. souř.

$\alpha^{-1} = (id)_{\alpha\varepsilon}$ ... matice transformace z báze $\varepsilon = ((1,0,0), (0,1,0), (0,0,1))$ do báze α

$(id)_{\varepsilon\alpha}$ ... transf. z α do ε

$= \alpha =$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$--- \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 2 & -1 & -3 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 1 & -1 & -1 \\ 2 & -1 & -3 \\ -1 & 1 & 2 \end{matrix}}_{\alpha^{-1} = (id)_{\alpha\varepsilon}} \\ 0 & 0 & 1 & \end{array}\right)$$

$$A = (\varphi)_{\varepsilon\varepsilon} = (id)_{\varepsilon\alpha} (\varphi)_{\alpha\alpha} (id)_{\alpha\varepsilon}$$

$$= \underbrace{\begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{(\varphi)_{\varepsilon\alpha}} \begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & -3 \\ -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & -3 \\ -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -2 & -5 \\ 4 & -3 & -6 \\ 1 & 0 & -1 \end{pmatrix} = A$$

$$\begin{pmatrix} 4 & -2 & -5 \\ 4 & -3 & -6 \\ 1 & 0 & -1 \end{pmatrix} \overset{u_1}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} = \overset{u_2}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

Př. $\mathbb{R}^3$, uvažme bázi

$u_1 = (1,-1,1)$, $u_2 = (1,1,0)$, $u_3 = (2,1,1)$

lin. zobr. $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$

$\varphi(u_1) = u_2$, $\varphi(u_2) = u_3$, $\varphi(u_3) = u_1$

Najděte matici A, aby $\underbrace{\text{v souřadnicích}}_{}$ standartní báze $\varphi(x) = Ax$

Matice pro bázi u_1, u_2, u_3

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array} \begin{array}{c} \begin{array}{ccc} \varphi(u_1) & \varphi(u_2) & \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \varphi(a\cdot u_1 + b\cdot u_2 + \ldots$$

$\varphi(u_1) = 0\cdot u_1 + 1\cdot u_2 + 0\cdot u_3$

Chceme najít $\varphi(e_1)$ $\varphi(e_2)$ $\varphi(e_3)$

$\varphi(1,0,0)$ $\varphi(0,1,0)$ $\varphi(0,0,1)$

$$\left(\begin{array}{c|c} u_1 & \varphi(u_1) \\ u_2 & \varphi(u_2) \\ u_3 & \varphi(u_3) \end{array}\right) \sim \left(\begin{array}{c|c} u_1 & \varphi(u_1) \\ u_1+u_2 & \varphi(u_1)+\varphi(u_2) = \varphi(u_1+u_2) \end{array}\right) \sim \ldots \sim \left(\begin{array}{c|c} e_1 & \varphi(e_1) \\ e_2 & \varphi(e_2) \\ e_3 & \varphi(e_3) \end{array}\right)$$

vždy na pravé straně zobrazení vektoru z levé strany

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 1 \\ 0 & 3 & -1 & -1 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -2 & -3 & 0 \\ 0 & 0 & -1 & 5 & 6 & 1 \end{array}\right) \sim$$

$_2 + c.$

$_3)$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & -2 & 0 \\ 0 & 1 & 0 & -2 & -3 & 0 \\ 0 & 0 & 1 & -5 & -6 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 4 & 1 \\ 0 & 1 & 0 & -2 & -3 & 0 \\ 0 & 0 & 1 & -5 & -6 & -1 \end{array}\right) \begin{array}{l} \leftarrow \varphi(e_1) = \varphi(1,0,0) \\ \leftarrow \varphi(e_2) = \varphi(0,1,0) \\ \leftarrow \varphi(e_3) = \varphi(0,0,1) \end{array}$$

$$A = \begin{pmatrix} 4 & -2 & -5 \\ 4 & -3 & -6 \\ 1 & 0 & -1 \end{pmatrix} \qquad \underset{u_1}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} = \underset{u_2}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

$\sim_3)$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & -2 & 0 \\ 0 & 1 & 0 & -2 & -3 & 0 \\ 0 & 0 & 1 & -5 & -6 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 4 & 1 \\ 0 & 1 & 0 & -2 & -3 & 0 \\ 0 & 0 & 1 & -5 & -6 & -1 \end{array}\right) \begin{array}{l} \leftarrow \varphi(e_1) = \varphi(1,0,0) \\ \leftarrow \varphi(e_2) = \varphi(0,1,0) \\ \leftarrow \varphi(e_3) = \varphi(0,0,1) \end{array}$$

$$A = \begin{pmatrix} 4 & -2 & -5 \\ 4 & -3 & -6 \\ 1 & 0 & -1 \end{pmatrix} \qquad \underset{u_1}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} = \underset{u_2}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

λ je vlastním číslem l. zobr. φ, pokud $\exists x \neq 0$: $\varphi(x) = \lambda \cdot x$ ← vl. vektor

$\varphi(x) = Ax = \lambda x \quad / -\lambda x$

$Ax - \lambda x = 0$

$Ax - \lambda E x = (A - \lambda E) x = 0$

$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$A - \lambda E$ musí mít lin. závislé řádky, aby $\exists$ nenulové x

Př. Najděte vl. č. a vl. v. $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$

$$\varphi(x) = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 3 & -\lambda \end{vmatrix} = (2-\lambda)\cdot(-\lambda) - 1\cdot 3 = \lambda^2 - 2\lambda - 3 = 0 = (\lambda-3)(\lambda+1)$$

$\Rightarrow$ vl. č. $-1, 3$

vl. v. pro $\lambda = -1$

$$\begin{pmatrix} 2-(-1) & 1 \\ 3 & -(-1) \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\to$ vl. v. $(1, -3)$

vl. v. pro $\lambda = 3$:

$$\begin{pmatrix} 2-3 & 1 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \to \text{vl. v. } (1,1)$$

Př. Najděte vl. č. a vl. v.

$$\varphi(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$ nemá řešení nad $\mathbb{R}$

$\Rightarrow$ nemá vl. č.

Př. Najděte vl. č. a vl. v. matic $B = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ -1 & -\lambda & 3 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(-\lambda)(1-\lambda) + 3 - 3\cdot(2-\lambda) - (1-\lambda)\cdot(-1) = \overbrace{(-2\lambda+\lambda^2)(1-\lambda)}^{-\lambda^3+3\lambda^2-2\lambda} + 3 - 6 + 3\lambda + 1 - \lambda$$

$$= -\lambda^3 + 3\lambda^2 - 2 = -(\lambda-1)\cdot(\lambda^2 - 2\lambda - 2)$$

$-1^3 + 3\cdot 1^2 - 2 = -1 + 3 - 2 = 0$

$$\begin{aligned} &(-\lambda^3 + 3\lambda^2 - 2) : (\lambda - 1) = -\lambda^2 + 2\lambda + 2 \\ &\underline{-(-\lambda^3 + \lambda^2)} \\ &\quad 2\lambda^2 - 2 \\ &\quad \underline{-(2\lambda^2 - 2\lambda)} \\ &\qquad 2\lambda - 2 \\ &\qquad \underline{-2\lambda - 2} \\ &\qquad\quad 0 \end{aligned}$$

$D = 4 - 4\cdot(-2)$
$= 12 = 4\cdot 3$

$\lambda_{1,2} = \dfrac{2 \pm 2\sqrt{3}}{2}$

$\lambda_{1,2} = 1 \pm \sqrt{3}$

Vlastní čísla: $1, 1-\sqrt{3}, 1+\sqrt{3}$

$\lambda = 1$: $\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 3 \\ 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \to u = (1, -1, 0)$

$\lambda = 1-\sqrt{3}$: $\begin{pmatrix} 1+\sqrt{3} & 1 & 0 \\ -1 & 1-\sqrt{3} & 3 \\ 1 & 1 & \sqrt{3} \end{pmatrix} \sim \begin{pmatrix} 1+\sqrt{3} & 1 & 0 \\ 1 & 1 & \sqrt{3} \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1-\sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\to v = (1, -1-\sqrt{3}, 1)$

$\lambda = 1+\sqrt{3}$ . . . . $w = (1, -1+\sqrt{3}, -1)$