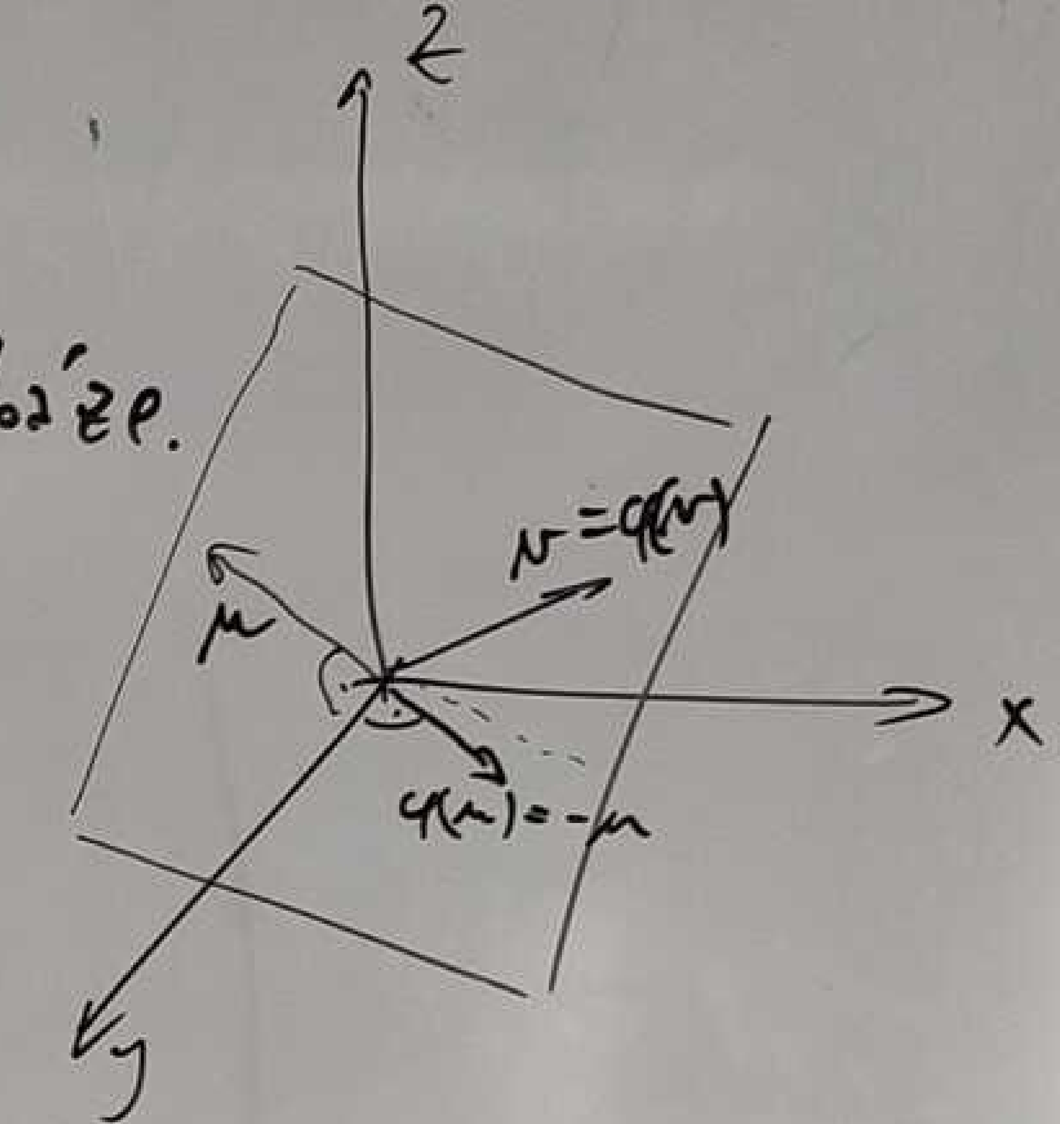
Př. $\varphi: \mathbb{R}^3 \to \mathbb{R}^3$

φ je symetrie podle roviny $x_2 + x_3 = 0$.

Najdete B, t.ž. $\varphi(x) = Bx$ v souř. standardní báze.



$u = (0,1,-1)$, $v = (1,0,0)$ $\qquad (0\ 1\ 1) \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0$

$w = (x,y,z) = (0,1,1)$ $\qquad (0\ 1\ 1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$

$x \perp y \Leftrightarrow \langle x, y \rangle = 0$

$\langle u, w \rangle = \langle (0,1,-1), (x,y,z) \rangle = 0 \cdot x + 1 \cdot y + (-1) \cdot z = y - z = 0$

$\langle v, w \rangle = \langle (1,0,0), (x,y,z) \rangle = 1 \cdot x + 0 \cdot y + 0 \cdot z = x = 0$

$\varphi(au + bv) = a\,\varphi(u) + b\,\varphi(v)$

$\varphi(u) = u$, $\varphi(v) = v$, $\varphi(w) = -w$

$B = (\varphi(e_1), \varphi(e_2), \varphi(e_3))$

$$\left(\begin{array}{c|c} u & \varphi(u) \\ v & \varphi(v) \\ w & \varphi(w) \end{array}\right) \sim \dots \sim \left(\begin{array}{c|c} e_1 & \varphi(e_1) \\ e_2 & \varphi(e_2) \\ e_3 & \varphi(e_3) \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right) \begin{array}{l} \leftarrow \varphi(e_1) \\ \leftarrow \varphi(e_2) \\ \leftarrow \varphi(e_3) \end{array}$$

$$\begin{array}{ccc} \varphi(e_1) & \varphi(e_2) & \varphi(e_3) \end{array}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Vl. čísla matice B: $1, -1$

$\varphi(x) = Bx = \lambda x$

Vl. vektor pro $\lambda = 1$: u, v

Vl. vektor pro $\lambda = -1$: w

Př. Najděte vl. č. a vektory matice $B = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ -1 & -\lambda & 3 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(-\lambda)(1-\lambda) + 3 - 3(2-\lambda) - (-1)(1-\lambda)$$

$$= \underbrace{(-\lambda)(2-3\lambda+\lambda^2)}_{-2\lambda+3\lambda^2-\lambda^3} + 3 - 6 + 3\lambda + 1 - \lambda$$

$$= -\lambda^3 + 3\lambda^2 - 2$$

$$-1^3 + 3\cdot 1^2 - 2 = -1 + 3 - 2 = 0$$

$$= -(\lambda - 1)\cdot(\lambda^2 - 2\lambda - 2) = 0$$

$$D = b^2 - 4ac = 4 - 4\cdot(-2)$$
$$= 4 + 8 = 12$$
$$\sqrt{D} = \sqrt{12} = \sqrt{4\cdot 3} = 2\sqrt{3}$$
$$\lambda_{1,2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

$\frac{p}{q}$ je kořen polynomu $a_m x^m + \ldots + a_1 x + a_0$

$p | a_0$, $q | a_m$

$$(\lambda^3 - 3\lambda^2 + 2) : (\lambda - 1) = \lambda^2 - 2\lambda - 2$$
$$\frac{-(\lambda^3 - \lambda^2)}{-2\lambda^2 + 2}$$
$$\frac{-(-2\lambda^2 + 2\lambda)}{-2\lambda + 2}$$
$$\frac{-(-2\lambda + 2)}{0}$$

vl. čísla: $\lambda = 1, 1+\sqrt{3}, 1-\sqrt{3}$

vl. v. pro $\lambda = 1$:

$$\begin{pmatrix} 2-1 & 1 & 0 \\ -1 & -1 & 3 \\ 1 & 1 & 1-1 \end{pmatrix} \cdot 2$$

$$\sim \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad -x-y+3z$$

$$u = (1, -1, 0)$$

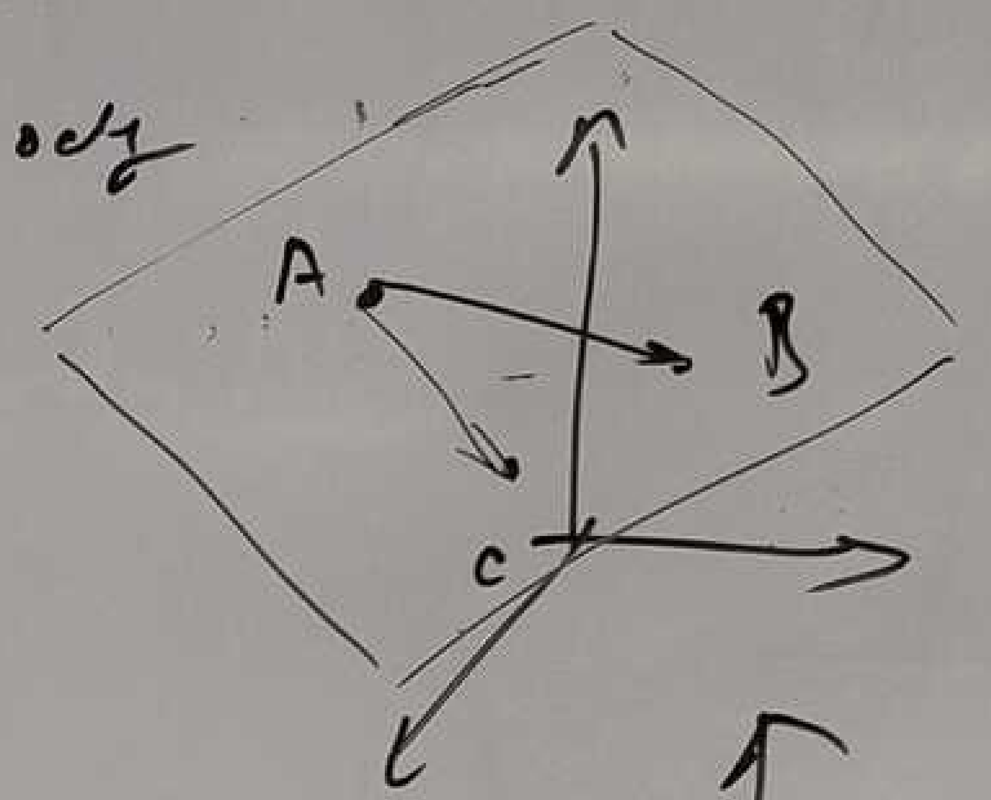
pro $\lambda = 1 + \sqrt{3}$

$$\begin{pmatrix} 1-\sqrt{3} & 1 & 0 \\ -1 & -1-\sqrt{3} & 3 \\ 1 & 1 & -\sqrt{3} \end{pmatrix} \sim \begin{pmatrix} 1-\sqrt{3} & 1 & 0 \\ 1 & 1 & -\sqrt{3} \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1+\sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v = (1, -1+\sqrt{3}, 1)$$

pro $\lambda = 1 - \sqrt{3}$ ... obdobně

Afinní podprostor v A_3: $\mathcal{M}: A + \underbrace{Z(\mathcal{M})}_{\text{vektorový prostor}} = A + a\cdot u + b\cdot v$, $a, b \in \mathbb{R}$



Př. Najděte nejmenší afinní podprostor obsahující body

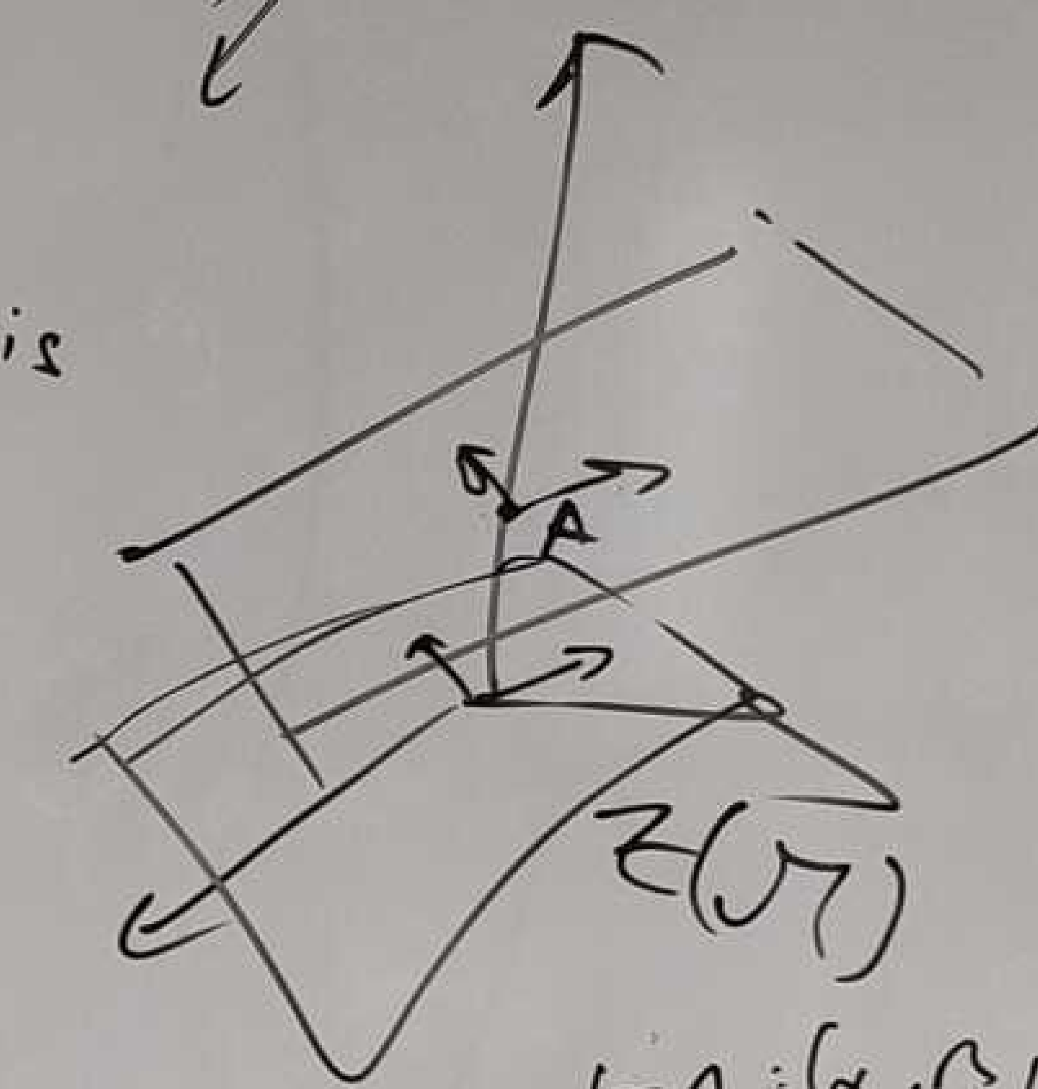
$A = [5,2,1]$, $B = [4,1,0]$, $C = [-3,1,0]$

$\mathcal{M}: A + Z(\mathcal{M}) = A + a\,\overrightarrow{AB} + b\,\overrightarrow{AC}$

$\overrightarrow{AB} = B - A = (-1,-1,-1)$, $\overrightarrow{AC} = (-8,-1,-1)$

$\mathcal{M}: [5,2,1] + \underbrace{a(-1,-1,-1) + b(-8,-1,-1)}_{Z(\mathcal{M})}$ .... parametrický popis

$\left(B = [5,2,1] + 1\cdot(-1,-1,-1) + 0\cdot(-8,-1,-1)\right)$



$\alpha x + \beta y + \gamma z = \delta$

$\alpha x + \beta y + \gamma z = 0$ ... impl. popis $Z(\mathcal{M})$

$(\alpha\ \beta\ \gamma)\cdot\begin{pmatrix}-1\\-1\\-1\end{pmatrix} = 0$

$(\alpha\ \beta\ \gamma)\begin{pmatrix}-8\\-1\\-1\end{pmatrix} = 0$

$$\begin{pmatrix}-1 & -1 & -1\\ -8 & -1 & -1\end{pmatrix} \sim \begin{pmatrix}1 & 1 & 1\\ 0 & 7 & 7\end{pmatrix} \sim \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 1\end{pmatrix} \sim \begin{array}{c}\begin{array}{ccc}\alpha & \beta & \gamma\end{array}\\ \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}\end{array}$$

$\gamma = t$
$\beta = -t$
$\alpha = 0$

$t = 1: (\alpha, \beta, \gamma) = (0, 1, -1)$

$\Downarrow$

$y - z = 0$ ... $Z(\mathcal{M})$

$y - z = \delta$ ... $\mathcal{M}$

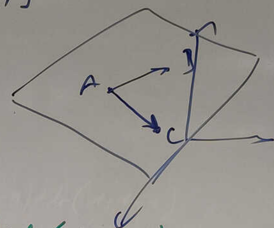
dosadím $[5,2,1]$: $2 - 1 = \delta$

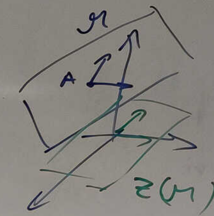
$\boxed{y - z = 1 \ \dots\ \mathcal{M}}$

STEJNÝ PŘÍKLAD, JAK NA MINULÉ STRANĚ

Př. Najděte nejmenší af. podpr. v A_3 obsahující body

$A=[5,2,1]$, $B=[4,1,0]$, $C=[-3,1,0]$





$\overrightarrow{AB} = B-A = (-1,-1,-1)$

$\overrightarrow{AC} = C-A = (-8,-1,-1)$

$\rho: A + a\cdot\overrightarrow{AB} + b\cdot\overrightarrow{AC} = [5,2,1] + a\cdot(-1,-1,-1) + b\cdot(-8,-1,-1)$ ... parametrický popis

$x = 5 - a - 8b$

$y = 2 - a - b$

$z = 1 - a - b$

$\left(B \in \rho : [4,1,0] = [5,2,1] + 1\cdot(-1,-1,-1) + 0\cdot(-8,-1,-1)\right)$

$\alpha x + \beta y + \gamma z = \delta$

dosadíme A: $5\alpha + 2\beta + \gamma - \delta = 0$

B: $4\alpha + \beta - \delta = 0$

C: $-3\alpha + \beta - \delta = 0$

$0x + 1\cdot y + (-1)\cdot z = 1$

$\boxed{y - z = 1}$

Alternativní postup

$$\begin{pmatrix} 5 & 2 & 1 & -1 \\ 4 & 1 & 0 & -1 \\ -3 & 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 4 & 1 & 0 & -1 \\ -3 & 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -3 & -4 & -1 \\ 0 & 4 & 3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 4 & 3 & -1 \end{pmatrix}$$

$$\xleftarrow{t=1} \quad \sim \begin{array}{c} \begin{matrix} \alpha & \beta & \gamma & \delta \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 7 & 7 \end{pmatrix} \end{array}$$

$\delta = t$

$7\gamma = -7\delta \rightarrow \gamma = -t$

$\beta = \gamma + 2\delta = t$

$\alpha = -\beta - \gamma = 0$

Př. Najděte průnik rovin

$$\mathcal{M}: \overset{A}{[2,3,4]} + a\,\overset{u}{(1,1,1)} + b\,\overset{v}{(0,0,1)}$$

$$\mathcal{N}: \overset{B}{[2,2,4]} + c\,\overset{w}{(1,0,1)} + d\,\overset{w'}{(2,0,1)}$$

$$x \in \mathcal{M} \cap \mathcal{N} \Leftrightarrow x \in \mathcal{M} \;\&\; x \in \mathcal{N}$$

$$x = A + a u + b v = B + c w + d w'$$

$$a u + b v - c w - d w' = B - A$$

$$\begin{array}{ccccc} a & b & c & d & B-A \\ \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 & 0 \\ 1 & 0 & -1 & -2 & 0 \end{array}\right) \sim \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & -2 & 1 \end{array}\right) \end{array}$$

$d = t$

$-c = 1 + 2d$

$c = -1 - 2t$

$b = 1 + c + d$

$= 1 - 1 - 2t + t$

$= -t$

$a = -1$

$$\mathcal{M} \cap \mathcal{N}: [2,3,4] + (-1)\cdot(1,1,1) + (-t)\cdot(0,0,1)$$

$$= [1,2,3] + t\,(0,0,-1)$$

Př. Určete vzájemnou polohu

$$\pi: \begin{aligned} 3x_1 + x_2 + 2x_3 &= 5 \\ 5x_1 - x_2 \quad\quad &= 3 \end{aligned} \qquad \rho: [-3,0,0] + a(3,1,2) + b(5,-1,0)$$

$$\pi \cap \rho: \begin{aligned} 3(-3+3a+5b) + (a-b) + 2\cdot 2a &= 5 \\ 5(-3+3a+5b) - (a-b) &= 3 \end{aligned}$$

$14a + 14b = 14 \quad /:14$

$14a + 26b = 18 \quad /:2$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 7 & 13 & 9 \end{array}\right) \sim \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 6 & 2 \end{array}\right) \end{array}$$

$6b = 2 \rightarrow b = \frac{1}{3}$

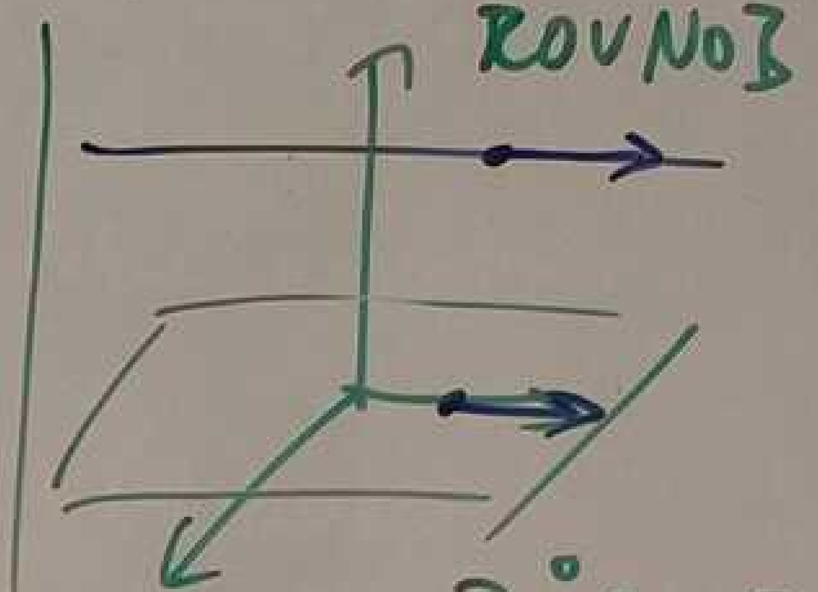
$a = 1 - b = \frac{2}{3}$

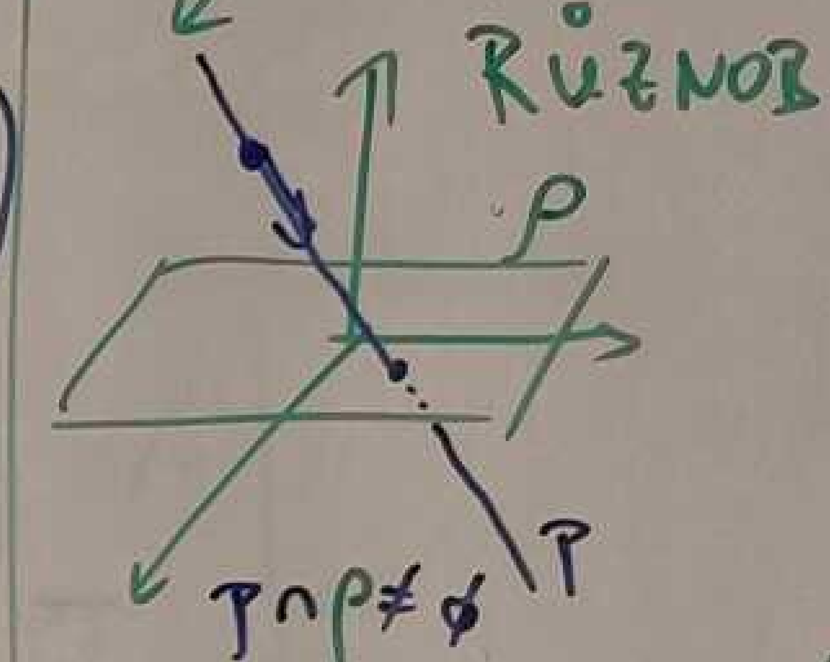
$\Rightarrow$ průnik je jeden bod

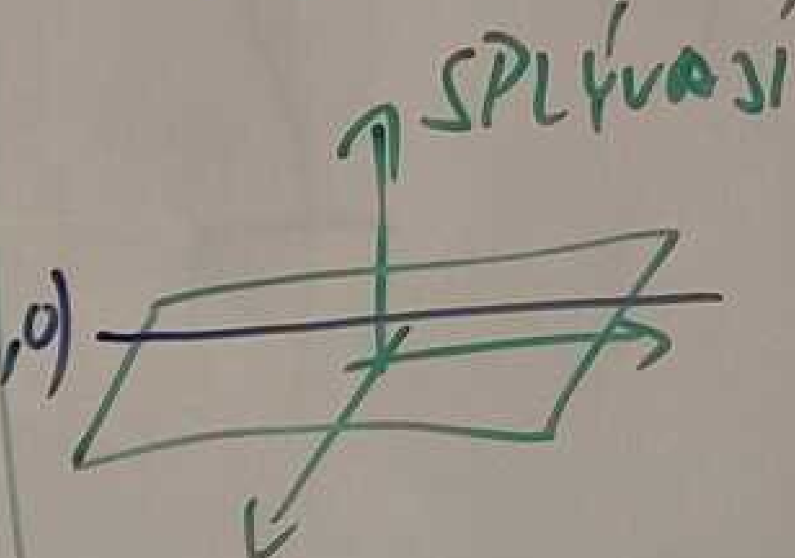
$$\pi \cap \rho = [-3,0,0] + \tfrac{2}{3}(3,1,2) + \tfrac{1}{3}(5,-1,0)$$

$$= \left[\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{4}{3}\right]$$

$\Rightarrow$ RŮZNOBĚŽNÉ







$\underline{\text{Př.}}$ Určete vzájemnou polohu roviny $\rho: \overset{O}{[3,-1,0]} + s\overset{u}{(-1,1,1)} + t\cdot\overset{v}{(2,1,0)}$

a přímky:

a) $p: \overset{P}{[7,4,2]} + a\overset{w}{(5,-2,-3)}$

b) $q: \overset{Q}{[1,2,3]} + b(1,5,3)$

c) $r: \overset{R}{[1,2,3]} + c(1,1,1)$

Potřebujeme rozdělit: $Z(p) \subseteq Z(\rho)$

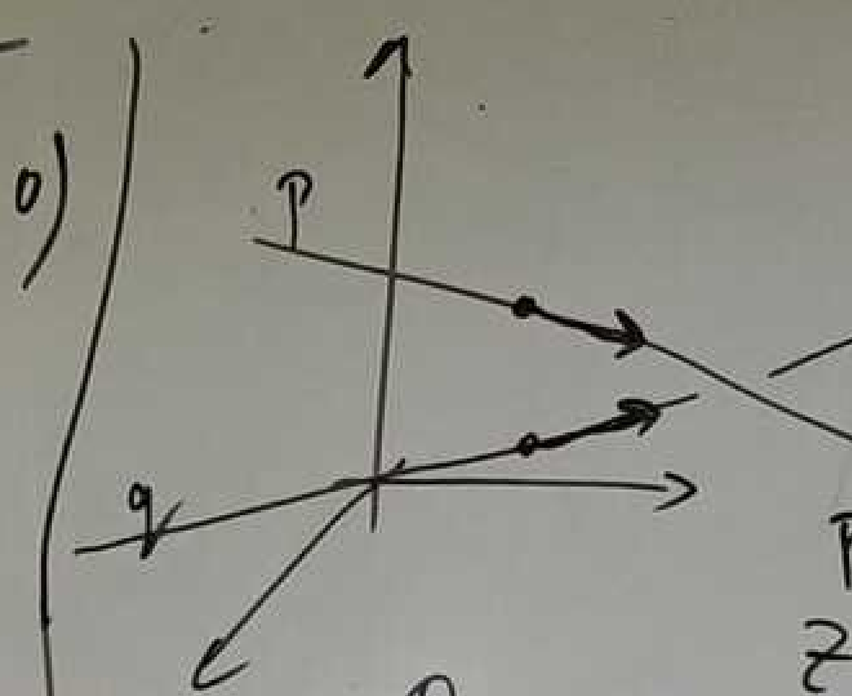
$p \cap \rho = \emptyset$?

a)
$$\begin{array}{c} \begin{array}{cccc} s & t & -a & P-O \end{array} \\ \left(\begin{array}{ccc|c} -1 & 2 & 5 & 4 \\ 1 & 1 & -2 & 5 \\ 1 & 0 & -3 & 2 \end{array}\right) \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ u & v & w \end{array} \end{array} \sim \left(\begin{array}{ccc|c} -1 & 2 & 5 & 4 \\ 0 & 3 & 3 & 9 \\ 0 & 2 & 2 & 6 \end{array}\right) \begin{array}{l} \\ /:3 \\ /:2 \end{array} \sim \begin{array}{c} \left(\begin{array}{ccc|c} \underline{-1} & 2 & 5 & 4 \\ 0 & \underline{1} & 1 & 3 \\ \cancel{0} & \cancel{1} & \cancel{1} & \cancel{3} \end{array}\right) \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ u & v & w \end{array} \end{array}$$

má řešení

$\Rightarrow$ w je LZ na u, v

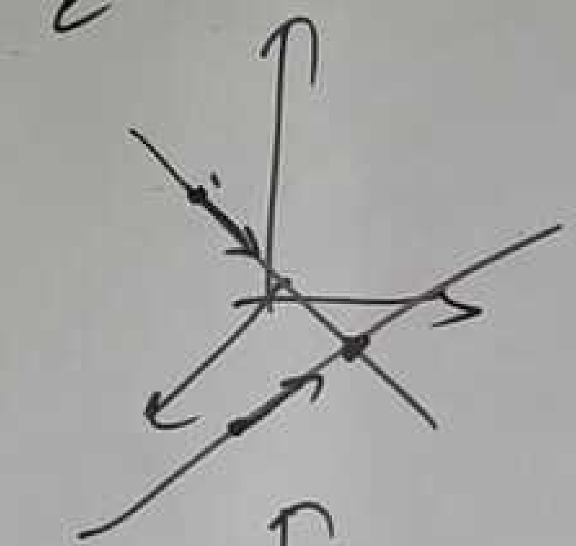
$\Rightarrow p \cap \rho \neq \emptyset,\ Z(p) \subseteq Z(\rho) \Rightarrow$ splývají



mimoběžné

$p \cap q = \emptyset$

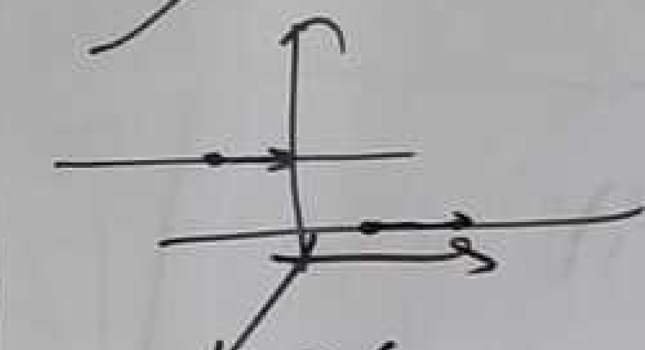
$Z(p) \not\subseteq Z(q)$



různoběžné

$p \cap q \neq \emptyset$

$Z(p) \cap Z(q) = \{o\}$



rovnoběžné

+ případ. splývají