

# Cross Entropy

PA154 Language Modeling (1.4)

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## Coding: Example

- How many bits do we need for ISO Latin 1?
  - ⇒ the trivial answer: 8
- Experience: some chars are more common, some (very) rare:
  - ...so what if we use more bits for the rare, and less bits for the frequent? (be careful: want to decode (easily)!)
    - suppose:  $p('a') = 0.3, p('b') = 0.3, p('c') = 0.3$ , the rest:  $p(x) \cong .0004$
    - code: 'a' ~ 00, 'b' ~ 01, 'c' ~ 10, rest:  $11b_1b_2b_3b_4b_5b_6b_7b_8$
    - code 'acbbébaac':
 

|    |    |    |    |                   |    |    |    |    |    |
|----|----|----|----|-------------------|----|----|----|----|----|
| 00 | 10 | 01 | 01 | <u>1100001111</u> | 10 | 01 | 00 | 00 | 10 |
| a  | c  | b  | b  | é                 | c  | b  | a  | a  | c  |
    - number of bits used: 28 (vs. 80 using "naive" coding)
  - code length  $\sim -\log(\text{probability})$

## Cross-Entropy

- Typical case: we've got series of observations
  $T = \{t_1, t_2, t_3, t_4, \dots, t_n\}$  (numbers, words, ...;  $t_1 \in \Omega$ ); estimate (sample):  $\forall y \in \Omega : \tilde{p}(y) = \frac{c(y)}{|T|}$ ,  
def.  $c(y) = |\{t \in T; t = y\}|$
- ...but the true  $p$  is unknown; every sample is too small!
- Natural question: how well do we do using  $\tilde{p}$  (instead of  $p$ )?
- Idea: simulate actual  $p$  by using a different  $T$  (or rather: by using different observation we simulate the insufficiency of  $T$  vs. some other data ("random" difference))

## "Coding" Interpretation of Entropy

- The least (average) number of bits needed to encode a message (string, sequence, series, ...) (each element having being a result of a random process with some distribution  $p$ ):  $= H(p)$
- Remember various compressing algorithms?
  - they do well on data with repeating (= easily predictable = = low entropy) patterns
  - their results though have high entropy  $\Rightarrow$  compressing compressed data does nothing

## Entropy of Language

- Imagine that we produce the next letter using

$$p(l_{n+1}|l_1, \dots, l_n),$$

where  $l_1, \dots, l_n$  is the sequence of **all** the letters which had been uttered so far (i.e.  $n$  is really big!); let's call  $l_1, \dots, l_n$  the **history**  $h(h_{n+1})$ , and all histories  $H$ :

- Then compute its entropy:
  - $-\sum_{h \in H} \sum_{l \in A} p(l, h) \log_2 p(l|h)$
- Not very practical, isn't it?

## Cross Entropy: The Formula

- $H_{p'}(\tilde{p}) = H(p') + D(p' || \tilde{p})$
- $H_{p'}(\tilde{p}) = -\sum_{x \in \Omega} p'(x) \log_2 \tilde{p}(x)$
- $p'$  is certainly not the true  $p$ , but we can consider it the "real world" distribution against which we test  $\tilde{p}$
- note on notation (confusing ...):  $\frac{p}{p'} \leftrightarrow \tilde{p}$ , also  $H_{T'}(p)$
- (Cross)Perplexity:  $G_{p'}(p) = G_{T'}(p) = 2^{H_{p'}(\tilde{p})}$

## Conditional Cross Entropy

- So far: "unconditional" distribution(s)  $p(x), p'(x)$ ...
- In practice: virtually always conditioning on context
- Interested in: sample space  $\Psi$ , r.v.  $Y, y \in \Psi$ ;  
context: sample space  $\Omega$ , r.v.  $X, x \in \Omega$ ;  
"our" distribution  $p(y|x)$ , test against  $p'(y, x)$ , which is taken from some independent data:

$$H_{p'}(p) = - \sum_{y \in \Psi, x \in \Omega} p'(y, x) \log_2 p(y|x)$$

## Sample Space vs. Data

- In practice, it is often inconvenient to sum over the space(s)  $\Psi, \Omega$  (especially for cross entropy!)
- Use the following formula:  $H_{p'}(p) = - \sum_{y \in \Psi, x \in \Omega} p'(y, x) \log_2 p(y|x) = -1/|T'| \sum_{i=1 \dots |T'|} \log_2 p(y_i|x_i)$
- This is in fact the normalized log probability of the "test" data:

$$H_{p'}(p) = -1/|T'| \log_2 \prod_{i=1 \dots |T'|} p(y_i|x_i)$$

## Computation Example

- $\Omega = \{a, b, \dots, z\}$ , prob. distribution (assumed/estimated from data):  $p(a) = .25, p(b) = .5, p(\alpha) = \frac{1}{64}$  for  $\alpha \in \{c..r\}, p(\alpha) = 0$  for the rest: s,t,u,v,w,x,y,z
- Data (test): barb  $p'(a) = p'(r) = .25, p'(b) = .5$
- Sum over  $\Omega$ :  

|                                |    |    |   |   |   |   |   |     |   |   |     |   |   |     |   |
|--------------------------------|----|----|---|---|---|---|---|-----|---|---|-----|---|---|-----|---|
| $\alpha$                       | a  | b  | c | d | e | f | g | ... | p | q | r   | s | t | ... | z |
| $-p'(\alpha) \log_2 p(\alpha)$ | .5 | .5 | 0 | 0 | 0 | 0 | 0 | 0   | 0 | 0 | 1.5 | 0 | 0 | 0   | 0 |
- Sum over data:  

|                  |     |     |     |     |      |
|------------------|-----|-----|-----|-----|------|
| $i / s_i$        | 1/b | 2/a | 3/r | 4/b |      |
| $-\log_2 p(s_i)$ | 1   | 2   | 6   | 1   | = 10 |

  
 $(1/4) \times 10 = 2.5$

## Cross Entropy: Some Observations

- $H(p)$  ??<, =, >??  $H_{p'}(p)$  : ALL!
- Previous example:  
 $p(a) = .25, p(b) = .5, p(\alpha) = \frac{1}{64}$  for  $\alpha \in \{c..r\}, = 0$  for the rest: s,t,u,v,w,x,y,z  
 $H(p) = 2.5 \text{ bits} = H(p')(\text{barb})$
- Other data: probable:  $(\frac{1}{8})(6 + 6 + 6 + 1 + 2 + 1 + 6 + 6) = 4.25$   
 $H(p) < 4.25 \text{ bits} = H(p')(\text{probable})$
- And finally: abba:  $(\frac{1}{4})(2 + 1 + 1 + 2) = 1.5$   
 $H(p) > 1.5 \text{ bits} = H(p')(\text{abba})$
- But what about: baby  $-p'('y') \log_2 p('y') = -.25 \log_2 0 = \infty$  (??)

## Cross Entropy: Usage

- Comparing data??
  - NO! (we believe that we test on real data!)
- Rather: comparing distributions (vs. real data)
- Have (got) 2 distributions:  $p$  and  $q$  (on some  $\Omega, X$ )
  - which is better?
  - better: has lower cross-entropy (perplexity) on real data  $S$
- "Real" data:  $S$

$$H_S(p) = -1/|S| \sum_{i=1 \dots |S|} \log_2 p(y_i|x_i) \quad H_S(q) = -1/|S| \sum_{i=1 \dots |S|} \log_2 q(y_i|x_i)$$

## Comparing Distributions

- $p(\cdot)$  from previous example:  $H_S(p) = 4.25$   
 $p(a) = .25, p(b) = .5, p(\alpha) = \frac{1}{64}$  for  $\alpha \in \{c..r\}, = 0$  for the rest: s,t,u,v,w,x,y,z
- $q(\cdot)$  (conditional; defined by a table):

|                        |   |    |   |   |   |      |   |       |
|------------------------|---|----|---|---|---|------|---|-------|
| $q(\cdot) \rightarrow$ | a | b  | e | l | o | p    | r | other |
| a                      | 0 | .5 | 0 | 0 | 0 | .125 | 0 | 0     |
| b                      | 1 | 0  | 0 | 0 | 1 | .125 | 0 | 0     |
| e                      | 0 | 0  | 0 | 1 | 0 | .125 | 0 | 0     |
| l                      | 0 | .5 | 0 | 0 | 0 | .125 | 0 | 0     |
| o                      | 0 | 0  | 0 | 0 | 0 | .125 | 1 | 0     |
| p                      | 0 | 0  | 0 | 0 | 0 | .125 | 0 | 1     |
| r                      | 0 | 0  | 0 | 0 | 0 | .125 | 0 | 0     |
| other                  | 0 | 0  | 1 | 0 | 0 | .125 | 0 | 0     |

$$(1/8) (\log(p|\text{oth.}) + \log(r|p) + \log(o|r) + \log(b|o) + \log(a|b) + \log(b|a) + \log(l|b) + \log(e|l))$$

$$(1/8) ( 0 + 3 + 0 + 0 + 1 + 0 + 1 + 0 )$$

$$H_S(q) = .625$$