Wir haben demnach folgenden Hilfssatz bewiesen:

Hilfssatz. *Sei* $M \subset I$ *eine perfekte nulldimensionale Menge,* $L \subset M$ *eine PCA-Menge; dann existiert eine Funktion* $H_1(x, y, z)$ *mit folgenden Eigenschaften:*

(a_1) $H_1(x, y, z)$ *ist für* $(x, y) \in I \times I$, $z \in M$ *stetig,*

(a_2) *aus* $(z_1 \neq z_2)$ *folgt* $[H_1(0, 0, z_1) \neq H_1(0, 0, z_2)]$,

(a_3) *aus* $(z \in L)$ *folgt* $[H_1(x, y, z) \in \Psi]$,

(a_4) *aus* $(z \in M - L)$ *folgt* $[H_1(x, y, z)$ *non* $\in \Psi]$.

Der Übergang von diesem Hilfssatz zum Satz erfolgt nun in genau derselben Weise, wie in meiner unter [3]) zitierten Arbeit (S. 248—249).

Żegiestów, 3/IX 1936.

Formal definitions in the theory of ordinal numbers [1]).

By

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1. The purpose of this paper is to extend to the transfinite ordinals, and functions of transfinite ordinals, the theory of formal definition which has been developed by one of the present authors [2]) for functions of positive integers. The notation and terminology are those previously employed by the authors [3]).

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[1]) Presented to the American Mathematical Society, September 1935.

[2]) S. C. Kleene, *A theory of positive integers in formal logic*, Amer. Jour. Math., vol. 57, 1935, pp. 153—173, 219—244. It should be observed that the part of this paper of Kleene which is concerned with problems of formal definition depends only on Church's rules of procedure I—III and is therefore unaffected by the fact that the complete system of Church, making use of the properties of Π and &, is now known to lead to contradiction (S. C. Kleene and J. B. Rosser, *The inconsistency of certain formal logics*, Ann. Math., vol. 36, 1935, pp. 630—636). Indeed these problems of formal definition have an interest which is independent even of the question whether there exists a consistent, adequate system of symbolic logic which embodies the rules of procedure I—III. The fact of this independent interest is made especially clear by the results of S. C. Kleene, loc. cit., § 18, and of Alonzo Church, *An unsolvable problem of elementary number theory*, forthcoming (abstract in Bull. Amer. Math. Soc., vol. 41, 1935, p. 332). Also it is thought that the considerations of the present paper have a bearing on the distinction between constructive and non-constructive ordinals, and on questions of enumerability and effective enumerability.

[3]) See Alonzo Church, *A set of postulates for the foundation of logic*, Ann. Math., vol. 33, 1932; pp. 346—366; S. C. Kleene, *loc. cit.*, and *Proof by cases in formal logic*, Ann. Math., vol. 35, 1934, pp. 529—544; Alonzo Church and J. B. Rosser, *Some properties of conversion*, forthcoming (abstract in Bull. Amer. Math. Soc., vol. 41, 1935, p. 332). These papers will hereafter be referred to by author or by author and date.

Note particularly the definition of *well-formed* (Kleene 1934 § 1), the abbreviations of well-formed formulas (Church 1932 § 6 and Kleene 1934 § 3), the use of heavy-type letters (Kleene 1934 § 3, (1) and (2), and 1935, footnotes

For the convenience of the reader we give here a brief explanation of this notation and terminology.

Consider the infinite list of symbols: {, }, (,), λ, [,], a, b, c, ..., of which $a, b, c, \ldots$ are to be called *proper symbols*. A formula is any finite sequence of symbols from this list.

A formula is called *well-formed*, and an occurrence of a proper symbol in a formula is said to be an occurrence as a *free* symbol in accordance with the following rules, and those only: (1) a formula consisting of a single proper symbol x is well-formed, and the occurrence of x in the formula x is an occurrence as a free symbol, (2) if $\mathbf{F}$ and $\mathbf{X}$ are well-formed, $\{\mathbf{F}\}(\mathbf{X})$ is well-formed, and the occurrences of proper symbols as free symbols in $\mathbf{F}$ and in $\mathbf{X}$ are occurrences as free symbols in $\{\mathbf{F}\}(\mathbf{X})$, (3) if $\mathbf{M}$ is well-formed, and x is a proper symbol which occurs as a free symbol in $\mathbf{M}$, $\lambda x[\mathbf{M}]$ is well-formed, and the occurrences of proper symbols other than x as free symbols in $\mathbf{M}$ are occurrences as free symbols in $\lambda x[\mathbf{M}]$. An occurrence of a proper symbol in a formula which is not an occurrence as a free symbol is called an occurrence as a *bound* symbol. By a free (bound) symbol of $\mathbf{K}$, is meant a proper symbol which occurs as a free (bound) symbol in $\mathbf{K}$.

Heavy type letters are used to represent undetermined well-formed formulas. The expression $\mathrm{S}^{x}_{\mathbf{N}}\mathbf{M}|$ is used to stand for the result of substituting $\mathbf{N}$ for x throughout $\mathbf{M}$.

An *immediate conversion* is any one of the following operations on well-formed formulas:

I. To replace any part $\lambda x[\mathbf{M}]$ by $\lambda y[\mathrm{S}^{x}_{y}\mathbf{M}|]$, where y is any proper symbol which does not occur in $\mathbf{M}$.

II. To replace any part $\{\lambda x[\mathbf{M}]\}(\mathbf{N})$ by $\mathrm{S}^{x}_{\mathbf{N}}\mathbf{M}|$, provided that the bound symbols of $\mathbf{M}$ are distinct both from x and from the free symbols of $\mathbf{N}$.

III. To replace any part $\mathrm{S}^{x}_{\mathbf{N}}\mathbf{M}|$ (other than a proper symbol which immediately follows a λ) by $\{\lambda x[\mathbf{M}]\}(\mathbf{N})$, provided that the bound symbols of $\mathbf{M}$ are distinct both from x and from the free symbols of $\mathbf{N}$.

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on pp. 219—220), the rules of procedure I—III (Church 1932, pp. 350, 355—356, and Kleene 1934 § 1) and the definition of *conversion* (Church, 1932, p. 357, and Kleene 1934, p. 535), the definition of *normal form* (Kleene, 1934, p. 535), the definitions of I (Kleene 1934, p. 536) and of 1, 2, 3, 4, ..., S (Kleene, 1935, p. 156), and the definition of *formal definability* (Kleene 1935, p. 219).

A finite sequence of immediate conversions is called a *conversion*; and it is said that $\mathbf{A}$ is convertible into $\mathbf{B}$, or $\mathbf{A}$ conv $\mathbf{B}$, if $\mathbf{B}$ is obtainable from $\mathbf{A}$ by a conversion.

A formula is said to be in *normal form* if it is well-formed and has no part of the form $\{\lambda x[\mathbf{M}]\}(\mathbf{N})$. A formula $\mathbf{B}$ is said to be a normal form of a formula $\mathbf{A}$ if $\mathbf{B}$ is in normal form and $\mathbf{A}$ conv $\mathbf{B}$.

As a matter of convenience in the actual writing down of formulas, various abbreviations are employed. $\{\ldots\{\{\mathbf{F}\}(\mathbf{X}_1)\}(\mathbf{X}_2)\ldots\}(\mathbf{X}_n)$ is written as

$$\{\mathbf{F}\}(\mathbf{X}_1, \ldots, \mathbf{X}_n) \quad \text{or} \quad \mathbf{F}(\mathbf{X}_1, \ldots, \mathbf{X}_n);$$

and

$$\lambda x_1[\lambda x_2[\ldots \lambda x_n[\mathbf{M}]\ldots]] \quad \text{as} \quad \lambda x_1 x_2 \ldots x_n.\mathbf{M}.$$

We use $I, S, 1, 2, 3, \ldots$ as abbreviations respectively for

$$\lambda x{\cdot}x, \quad \lambda nfx{\cdot}f(n(f,x)), \quad \lambda fx{\cdot}f(x), \quad \lambda fx{\cdot}f(f(x)), \quad \lambda fx{\cdot}f(f(f(x))), \ldots .$$

The formulas 1, 2, 3, ... are taken to represent the positive integers.

2. Using an arrow to mean "stands for" or "is an abbreviation for", we let:

$$0_o \to \lambda m{\cdot}m(1)$$
$$S_o \to \lambda am{\cdot}m(2, a).$$
$$L \to \lambda arm{\cdot}m(3, a, r).$$
$$1_o \to S_o(0_o), \qquad 2_o \to S_o(1_o), \qquad \text{etc.}$$

The formulas $0_o, 1_o, 2_o, \ldots$ are taken to represent the finite ordinals.

Here we are using the subscript o to distinguish notations used in connection with the present theory from like notations used in other connections. When the context precludes ambiguity, we may omit this subscript o, writing for instance $\omega+2$ instead of $\omega +_o 2_o$ (cf. § 4 below).

We adopt the following rules for the assignment of formulas to represent particular ordinals of the first and second number classes:

i. If $\mathbf{a}$ represents the ordinal a, and $\mathbf{b}$ conv $\mathbf{a}$, then $\mathbf{b}$ also represents a.

ii. 0_o represents the ordinal zero.

iii. If $\mathbf{a}$ represents the ordinal a, then $S_o(\mathbf{a})$ represents the successor of a.

iv. If b is the limit of an increasing sequence (series) of ordinals, $b_0, b_1, b_2, \ldots,$ of order type ω, and if $\boldsymbol{r}$ is a formula such that the formulas $\boldsymbol{r}(0_o), \boldsymbol{r}(1_o), \boldsymbol{r}(2_o), \ldots$ represent the ordinals $b_0, b_1, b_2, \ldots,$ respectively, then $L(0_o, \boldsymbol{r})$ represents b.

Since one of the formulas assigned by these rules to represent the least ordinal ω of the second number class is $L(0_o, I)$, we let,

$$\omega \rightarrow L(0_o, I).$$

We shall call an ordinal of the first or second number class *formally definable*, or *λ-definable*, if there is, under Rules i—iv, at least one formula assigned to represent it.

From the possibility under iv of using different sequences for any particular b (or of using different $\boldsymbol{r}$'s for any particular sequence) it follows that every formally definable ordinal of the second number class has an infinite number of non-interconvertible formulas assigned to represent it.

We shall say that a sequence of ordinals of order type ω is *formally defined as a function of ordinals* by $\boldsymbol{r}$ if, for every formula $\boldsymbol{a}$ which represents a finite ordinal a, $\boldsymbol{r}(\boldsymbol{a})$ represents the $(1+a)$th ordinal of the sequence.

It is clear that the question what ordinals of the second number class are formally definable is tied up with the question what sequences of ordinals are formally definable as functions of ordinals, in such a way that neither question can be regarded as prior to the other.

From Theorem 2 below and the enumerability of the set of all well-formed formulas, it follows (by a non-constructive argument) that the set of all formally definable ordinals is enumerable, and hence that there is a least ordinal ξ in the second number class which is not formally definable. It is then readily proved that no ordinal in the second class greater than ξ is formally definable.

It is to be emphasized, however, that the definition of ξ which has just been given is not constructive, in any usual sense of that term, and that no means is known to the authors of obtaining constructively an ordinal of the second number class which is not formally definable. In particular, it will be shown below that, if a and b are formally definable ordinals out of the first and second number classes, then the sum of a and b, the product of a and b, and the result of raising a to the power b are formally definable, and hence all ordinals less than ε_0 are formally definable. Moreover, if a is formally definable, then ε_a is formally definable, and hence it follows that all ordinals less than the least solution of $\varepsilon_x = x$ are formally definable. Similar theorems can be established in which various generalizations of the ε-numbers [4]) appear. Thus the boundary of ordinals in the second number class known to be formally definable is continually pushed upwards, and no constructively obtainable limit to this process is seen.

We prove the two following theorems by transfinite induction.

Theorem 1. *Every formula which represents an ordinal number under Rules* i—iv *has a normal form.*

Any formula which represents the ordinal zero must be convertible into 0_o and hence has the normal form 0_o. Suppose that every formula which represents an ordinal less than b has a normal form. If b is the successor of an ordinal a, any formula which represents b must be convertible into $S_o(\boldsymbol{a})$, where $\boldsymbol{a}$ represents a, and hence must have the normal form $\lambda m\cdot m(2, \boldsymbol{A})$, where $\boldsymbol{A}$ is the normal form of $\boldsymbol{a}$. If b is the limit of an increasing sequence of ordinals of order type ω, any formula which represents b must be convertible into $L(0_o, \boldsymbol{r})$, where there is some increasing sequence of ordinals of order type ω which has b as its limit and which is formally defined as a function of ordinals by $\boldsymbol{r}$; moreover $\boldsymbol{r}$ must have a normal form $\boldsymbol{R}$, because otherwise $\boldsymbol{r}(0_o)$ would lack a normal form [5]), and hence $L(0_o, \boldsymbol{r})$ must have the normal form $\lambda m\cdot m(3, 0_o, \boldsymbol{R})$.

Theorem 2. *If, under Rules* i—iv, *a formula* $\boldsymbol{b}$ *represents an ordinal* b, *then* $\boldsymbol{b}$ *cannot represent an ordinal distinct from* b.

The theorem is true if b is zero, because $\boldsymbol{b}$ then has $\lambda m\cdot m(1)$ as a normal form, and formulas which represent ordinals distinct from zero have a normal form of one of the forms, $\lambda m\cdot m(2, \boldsymbol{A})$, or $\lambda m\cdot m(3, 0_o, \boldsymbol{R})$, and hence cannot have $\lambda m\cdot m(1)$ as a normal form [6]). We proceed by induction with respect to b.

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[4]) Cf. O. Veblen, *Continuous increasing functions of finite and transfinite ordinals*, Trans. Amer. Math. Soc., vol. 9, 1908, pp. 280—292. All the particular ordinals defined by Veblen in this paper are formally definable in our present sense, including the ordinals E(1), E(2), ..., and even the first (second, and so on) solutions of the equation $f(1_1, \ldots, 1_\alpha) = \alpha$, page 292.

[5]) Church and Rosser, Thm. 2 Cor.

[6]) Church and Rosser, Thm. 1 Cor. 2.

If b is the successor of an ordinal a, then b is not the limit of any increasing sequence of ordinals and is not the successor of any ordinal other than a, and consequently $\boldsymbol{b}$ must have a normal form $\lambda m{\cdot}m(2, \boldsymbol{A})$, where $\boldsymbol{A}$ represents a. Therefore [6]) every normal form of b must be of the form $\lambda \boldsymbol{n}{\cdot}\boldsymbol{n}(\boldsymbol{2}, \boldsymbol{A}')$, where $\boldsymbol{2}$ conv 2 and $\boldsymbol{A}'$ conv $\boldsymbol{A}$ (being the same formulas except for possible alphabetical differences of bound symbols). Therefore $\boldsymbol{b}$ can represent only a successor of an ordinal represented by $\boldsymbol{A}$. By hypothesis of induction, $\boldsymbol{A}$ cannot represent an ordinal distinct from a. Therefore $\boldsymbol{b}$ cannot represent an ordinal distinct from b.

If b is the limit of an increasing sequence of ordinals of order type ω, then b is not the successor of any ordinal, and consequently $\boldsymbol{b}$ must have a normal form $\lambda m{\cdot}m(3, 0_o, \boldsymbol{R})$, where $\boldsymbol{R}$ formally defines as a function of ordinals an increasing sequence of ordinals of order type ω which has b as its limit. Therefore [6]) every normal form of $\boldsymbol{b}$ must be of the form $\lambda \boldsymbol{n}{\cdot}\boldsymbol{n}(\boldsymbol{3}, \boldsymbol{O}, \boldsymbol{R}')$, where $\boldsymbol{3}$ conv 3, $\boldsymbol{O}$ conv 0_o, $\boldsymbol{R}'$ conv $\boldsymbol{R}$. Therefore $\boldsymbol{b}$ can represent only a limit of an increasing sequence of ordinals which is formally defined as a function of ordinals by $\boldsymbol{R}$. But it follows from the hypothesis of induction that $\boldsymbol{R}$ can define only one increasing sequence of ordinals of order type ω. Therefore $\boldsymbol{b}$ can represent only b.

3. We shall call a function a *function in the first and second number classes* if the range of the independent variable (or of each independent variable) consists of the first and second number classes and the range of the dependent variable is contained in the first and second number classes.

We shall say that a function f, of n variables, in the first and second number classes, is *formally defined* by a formula $\boldsymbol{f}$ if, for every set of n formulas $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ which represent ordinals $x_1, x_2, \ldots, x_n$, respectively, in the first and second number classes, $\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$ is a formula which represents the ordinal $f(x_1, x_2, \ldots, x_n)$. We shall call a function f in the first and second number classes *formally definable*, or *λ-definable*, if there is at least one formula $\boldsymbol{f}$ by which it is formally defined.

We remark that if a formula $\boldsymbol{f}$ is to define (formally) a function, of n variables, in the first and second number classes, not only must $\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$ represent an ordinal in the first and second number classes for every set of n formulas $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ which represent ordinals in the first and second number classes, but also, if $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n$ are formulas which represent ordinals equal respectively to the ordinals represented by $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$, then $\boldsymbol{f}(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n)$ must represent an ordinal equal to the ordinal represented by $\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n)$.

It is clear that any function in the first and second number classes which is formally definable must be in a certain sense constructive, because given a formal definition of the function, the process of reduction to normal form provides an algorithm for calculating the values of the function (provided that we consider an ordinal to have been calculated if a formula in normal form which represents it has been obtained). Therefore certain functions which are not constructive in this sense are clearly not formally definable; in particular, the function f, such that $f(x, y)$ is equal to the ordinal zero or the ordinal one according to whether the ordinals x and y are equal or distinct, is not formally definable [7]).

It is conjectured, however, that every constructive function in the first and second number classes is formally definable. This conjecture is vague because of the lack of a satisfactory definition of the notion of a constructive function of ordinals, but it is supported by analogy with the case of functions of positive integers, where it is possible to give a plausible definition of constructivity and to prove the equivalence of constructivity and λ-definability [8]).

4. We proceed now to the proof of a theorem which is useful in many particular cases where it is required to obtain a formal definition of some given function in the first and second number classes.

Theorem 3. *If* $\boldsymbol{A}$, $\boldsymbol{G}$ *and* $\boldsymbol{H}$ *are given formulas having no free symbols, it is possible to find a set of eight formulas* $\boldsymbol{f}_{ijk}$ *(where the subscripts* i, j, k *take the values* 1 *and* 2) *satisfying the following conditions*:

$$\boldsymbol{f}_{1jk}(0_o) \text{ conv } \boldsymbol{A} \qquad \boldsymbol{f}_{2jk}(0_o) \text{ conv } \boldsymbol{A}(\boldsymbol{f}_{2jk})$$
$$\boldsymbol{f}_{i1k}(S_o(\boldsymbol{a})) \text{ conv } \boldsymbol{G}(\boldsymbol{a}) \qquad \boldsymbol{f}_{i2k}(S_o(\boldsymbol{a})) \text{ conv } \boldsymbol{G}(\boldsymbol{f}_{i2k}, \boldsymbol{a})$$
$$\boldsymbol{f}_{ij1}(L(\boldsymbol{a}, \boldsymbol{r})) \text{ conv } \boldsymbol{H}(\boldsymbol{a}, \boldsymbol{r}) \qquad \boldsymbol{f}_{ij2}(L(\boldsymbol{a}, \boldsymbol{r})) \text{ conv } \boldsymbol{H}(\boldsymbol{f}_{ij2}, \boldsymbol{a}, \boldsymbol{r}).$$

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[7]) The full proof of this assertion makes use of the results of Church, *An unsolvable problem of elementary number theory.*

[8]) See Church, *An unsolvable problem of elementary number theory*, and S. C. Kleene, *λ-definability and recursiveness*, forthcoming (abstract in Bull. Amer. Math. Soc., Vol. 41, 1935, No. 7).

This theorem is to be understood to mean, not merely that the formulas f_{ijk} "exist", but that means are at hand by which they can be effectively obtained in any given case.

In order to construct the formulas f_{ijk} we proceed as follows. We let,

$$\mathfrak{E}_1 \rightarrow \lambda fb \cdot b(4, f),$$
$$\mathfrak{E}_2 \rightarrow \lambda fb \cdot f(\lambda x \cdot x(\lambda n \cdot b(n, b))).$$

Using Kleene 1935, 15 III, we construct eight formulas $\boldsymbol{B}_{ijk}$ such that $\boldsymbol{B}_{ijk}(1) \operatorname{conv} \mathfrak{E}_i(\boldsymbol{A})$, $\boldsymbol{B}_{ijk}(2) \operatorname{conv} \mathfrak{E}_j(\boldsymbol{G})$, $\boldsymbol{B}_{ijk}(3) \operatorname{conv} \mathfrak{E}_k(\boldsymbol{H})$, and $\boldsymbol{B}_{ijk}(4) \operatorname{conv} I$. Then we let,

$$f_{ijk} \rightarrow \lambda x \cdot x(\lambda n \cdot \boldsymbol{B}_{ijk}(n, \boldsymbol{B}_{ijk})).$$

The proof that the formulas f_{ijk} have the required conversion properties is then straightforward.

In particular, we can, according to Theorem 3, obtain a formula $\mathfrak{f}$ such that $\mathfrak{f}(0_o) \operatorname{conv} I$, $\mathfrak{f}(S_o(\boldsymbol{a})) \operatorname{conv} \lambda x \cdot S_o(\mathfrak{f}(\boldsymbol{a}, x))$, and $\mathfrak{f}(L(\boldsymbol{a}, \boldsymbol{r})) \operatorname{conv} \lambda x \cdot L(\boldsymbol{a}, \lambda m \cdot \mathfrak{f}(\boldsymbol{r}(m), x))$. Then it can be proved, by transfinite induction, that, if $\boldsymbol{a}$ and $\boldsymbol{b}$ represent ordinals a and b, respectively, in the first and second number classes, $\mathfrak{f}(\boldsymbol{a}, \boldsymbol{b})$ represents the sum of b and a. Hence we let,

$$[\boldsymbol{b}] +_o [\boldsymbol{a}] \rightarrow \mathfrak{f}(\boldsymbol{a}, \boldsymbol{b}).$$

Likewise we may obtain a formula $\mathfrak{f}$ such that $\mathfrak{f}(0_o) \operatorname{conv} I$, $\mathfrak{f}(S_o(\boldsymbol{a})) \operatorname{conv} \mathfrak{f}(\boldsymbol{a})$, and $\mathfrak{f}(L(\boldsymbol{a}, \boldsymbol{r})) \operatorname{conv} \mathfrak{f}(\boldsymbol{a}, \mathfrak{f}(\boldsymbol{r}(0_o)))$. Then, by transfinite induction, if $\boldsymbol{a}$ represents an ordinal, $\mathfrak{f}(\boldsymbol{a}) \operatorname{conv} I$. Hence a constancy function, K_o, of ordinals may be defined by letting,

$$K_o \rightarrow \lambda xy \cdot \mathfrak{f}(y, x).$$

If $\boldsymbol{b}$ represents an ordinal, $K_o(\boldsymbol{a}, \boldsymbol{b}) \operatorname{conv} \boldsymbol{a}$.

Likewise we may obtain a formula $\mathfrak{f}$ such that $\mathfrak{f}(0_o) \operatorname{conv} K_o(0_o)$, $\mathfrak{f}(S_o(\boldsymbol{a})) \operatorname{conv} \lambda x \cdot \mathfrak{f}(\boldsymbol{a}, x) +_o x$, and $\mathfrak{f}(L(\boldsymbol{a}, \boldsymbol{r})) \operatorname{conv} \lambda x \cdot L(\boldsymbol{a}, \lambda m \cdot \mathfrak{f}(\boldsymbol{r}(m), x))$. Then, if $\boldsymbol{a}$ and $\boldsymbol{b}$ represent ordinals a and b respectively, $\mathfrak{f}(\boldsymbol{a}, \boldsymbol{b})$ represents the product of a and b (with a as multiplier and b as multiplicand). Hence we let,

$$[\boldsymbol{a}] \times_o [\boldsymbol{b}] \rightarrow \mathfrak{f}(\boldsymbol{a}, \boldsymbol{b})$$

The expression $[\boldsymbol{a}] \times_o [\boldsymbol{b}]$ may be abbreviated as $[\boldsymbol{a}][\boldsymbol{b}]$, when no ambiguity arises thereby.

Similarly, we may obtain a formula $\mathfrak{f}$ such that $\mathfrak{f}(0_o) \operatorname{conv} K_o(1_o)$, $\mathfrak{f}(S_o(\boldsymbol{a})) \operatorname{conv} \lambda x \cdot x \times_o \mathfrak{f}(\boldsymbol{a}, x)$, and $\mathfrak{f}(L(\boldsymbol{a}, \boldsymbol{r})) \operatorname{conv} \lambda x \cdot L(\boldsymbol{a}, \lambda m \cdot \mathfrak{f}(\boldsymbol{r}(m), x))$.

Then, if $\boldsymbol{a}$ and $\boldsymbol{b}$ represent ordinals a and b, $\mathfrak{f}(\boldsymbol{a}, \boldsymbol{b})$ represents the result of raising b to the power a. Hence we let,

$$\exp_o \rightarrow \lambda xy \cdot \mathfrak{f}(y, x)$$

and in any case where no confusion with exponentiation of positive integers, or other forms of exponentiation, is produced, we abbreviate $\exp_o(\boldsymbol{b}, \boldsymbol{a})$ as $[\boldsymbol{b}]^{[\boldsymbol{a}]}$.

A predecessor function of ordinals is formally defined by a formula P_o chosen to satisfy the relations [9] $P_o(0_o) \operatorname{conv} 0_o$, $P_o(S_o(\boldsymbol{a})) \operatorname{conv} \boldsymbol{a}$, and $P_o(L(\boldsymbol{a}, \boldsymbol{r})) \operatorname{conv} L(\boldsymbol{a}, \boldsymbol{r})$.

Let $\mathfrak{F}$ be so chosen that $\mathfrak{F}(0_o) \operatorname{conv} 0_o$, $\mathfrak{F}(S_o(\boldsymbol{a})) \operatorname{conv} \mathfrak{F}(\boldsymbol{a})$, and $\mathfrak{F}(L(\boldsymbol{a}, \boldsymbol{r})) \operatorname{conv} K_o(1_o, L(\boldsymbol{a}, \boldsymbol{r}))$. Then, if $\boldsymbol{a}$ represents an ordinal a, $\mathfrak{F}(\boldsymbol{a})$ is convertible into 0_o or 1_o according as a is finite or infinite.

Choose $\mathfrak{K}$ so that $\mathfrak{K}(0_o) \operatorname{conv} 0_o$, $\mathfrak{K}(S_o(\boldsymbol{a})) \operatorname{conv} K_o(1_o, \boldsymbol{a})$, and $\mathfrak{K}(L(\boldsymbol{a}, \boldsymbol{r})) \operatorname{conv} K_o(2_o, L(\boldsymbol{a}, \boldsymbol{r}))$. Then $\mathfrak{K}$ formally defines the *kind* of an ordinal in the following fashion: If $\boldsymbol{a}$ represents an ordinal a, $\mathfrak{K}(\boldsymbol{a})$ is convertible into 0_o or 1_o or 2_o according as a is zero, the successor of some ordinal, or the limit of some increasing sequence of ordinals.

The formula $\lambda n \cdot P_o(n(S_o, 0_o))$ defines the nth (finite) ordinal number as a function of the nth positive integer. Inversely, if $\mathfrak{J}$ is so chosen that $\mathfrak{J}(0_o) \operatorname{conv} 1$, $\mathfrak{J}(S_o(\boldsymbol{a})) \operatorname{conv} S(\mathfrak{J}(\boldsymbol{a}))$, and $\mathfrak{J}(L(\boldsymbol{a}, \boldsymbol{r})) \operatorname{conv} K_o(1, L(\boldsymbol{a}, \boldsymbol{r}))$, then $\mathfrak{J}$ defines the nth positive integer as a function of the nth (finite) ordinal number. Also, if a is any ordinal number of the second kind, $\mathfrak{J}$ defines the nth positive integer as a function of the nth ordinal number in the set of ordinal numbers from a on. With the aid of these transformations between ordinal numbers and positive integers, much of the theory of formal definition of functions of positive integers (Kleene 1935) can be carried over at once to the ordinal numbers.

Choose $\mathfrak{e}$ so that

$$\mathfrak{e}(0_o) \operatorname{conv} L(0_o, \lambda n \cdot \mathfrak{J}(n, \exp_o(\omega), 0_o)),$$
$$\mathfrak{e}(S_o(\boldsymbol{a})) \operatorname{conv} L(0_o, \lambda n \cdot \mathfrak{J}(n, \exp_o(\omega), S_o(\mathfrak{e}(\boldsymbol{a})))),$$

and

$$\mathfrak{e}(L(\boldsymbol{a}, \boldsymbol{r})) \operatorname{conv} L(\boldsymbol{a}, \lambda n \cdot \mathfrak{e}(\boldsymbol{r}(n))).$$

[9]) The choice of the third relation was made somewhat arbitrarily. Any other of the form required by Theorem 3 might have been used instead.

Then if $\boldsymbol{a}$ represents an ordinal number a, the formula $\epsilon(\boldsymbol{a})$ (which we abbreviate as $\varepsilon_{\boldsymbol{a}}$) represents the $(1+a)$th epsilon number.

Similarly formal definitions may be obtained for various generalizations of the ε-numbers, using Theorem 3 and generalizations of Theorem 3.

5. We let

$$Z_1 \rightarrow \mathfrak{Q}(\lambda x \cdot x(I), I),$$
$$Z_2 \rightarrow \mathfrak{Q}(\lambda xy \cdot S(x) - y, I),$$

where $\mathfrak{Q}$ is the formula defined in the first paragraph of § 17, Kleene 1935. Then Z_1 and Z_2 define the sequences 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, ... and 1, 2, 1, 3, 2, 1, 4, 3, 2, 1, ... respectively, as functions of positive integers (cf. Kleene 1935, 17 I).

Using Kleene 1935, 15 III, we obtain a formula $\mathfrak{B}$ such that $\mathfrak{B}(1)$ conv $\lambda f x \cdot K_o(x, f(0_o, 1))$ and, for all formulas $\boldsymbol{n}$ which represent positive integers, $\mathfrak{B}(S(\boldsymbol{n}))$ conv $\lambda f x \cdot f(x, \boldsymbol{n})$. Then using Theorem 3 we obtain a formula enm such that enm (0_o) conv $\lambda n \cdot n(I, 0_o)$,

$$\text{enm}\,(S_o(\boldsymbol{a})) \text{ conv } \lambda n \cdot \mathfrak{B}(n, \text{enm}, \boldsymbol{a}),$$

and

$$\text{enm}(L(\boldsymbol{a}, \boldsymbol{r})) \text{ conv } \lambda n \cdot K_o(\text{enm}\,(\boldsymbol{r}(P_o(Z_1(n, S_o, 0_o))), \ Z_2(n)), \boldsymbol{a}).$$

The formula enm has the property that, if $\boldsymbol{a}$ represents an ordinal a of the first or the second number class, other than zero, the infinite sequence enm $(\boldsymbol{a}, 1)$, enm $(\boldsymbol{a}, 2)$ enm $(\boldsymbol{a}, 3)$, ... is an enumeration, with repetitions, of the ordinals less than a (i. e., every term of the sequence represents such an ordinal, and every such ordinal is represented by at least one term of the sequence).

It is not, however, true, if $\boldsymbol{b}$ represents an ordinal equal to a, and $\boldsymbol{n}$ represents a positive integer, that enm $(\boldsymbol{b}, \boldsymbol{n})$ must represent an ordinal equal to enm $(\boldsymbol{a}, \boldsymbol{n})$. There is no formula which has this property in addition to the property of enm just described.

6. The assignment of formulas to represent ordinals which was set up in § 2 can be extended to larger ordinals (not, however, to all ordinals) by replacing Rule iv by the following transfinite set of rules, iv$_a$, where the subscript a may take on as value any ordinal a formula to represent which has been previously assigned:

iv$_a$. If ι_a is the least ordinal a formula to represent which is not assigned by the rules i, ii, iii, iv$_i$ $(0 \leqq i < a)$, if b is the limit of an increasing sequence of ordinals of order type ι_a, which is formally defined as a function of ordinals by $\boldsymbol{r}$, and if $\boldsymbol{a}$ represents a, then $L(\boldsymbol{a}, \boldsymbol{r})$ represents b.

For the rule iv$_0$ we take ι_0 to be ω, so that Rule iv$_0$ is identical with Rule iv.

It can be proved as before that a formula which represents an ordinal must have a normal form, and that distinct ordinals cannot be represented by the same formula; and it will be found that the formal definitions of particular functions of ordinals which are obtained in § 4 remain valid under the extended assignment of formulas to represent ordinals.

The ordinals ι_a are defined formally by

$$\iota_a \rightarrow L(\boldsymbol{a}, I).$$

Moreover, using Theorem 3, we may find a formula $\mathfrak{o}$ such that $\mathfrak{o}(0_o)$ conv $L(0_o, I)$, $\mathfrak{o}(S_o(\boldsymbol{a}))$ conv $L(1_o +_o \boldsymbol{a}, I)$, and $\mathfrak{o}(L(\boldsymbol{a}, \boldsymbol{r}))$ conv $L(\boldsymbol{a}, \lambda n \cdot L(\boldsymbol{r}(n), I))$, and so obtain a set of ordinals ω_a defined formally by

$$\omega_a \rightarrow \mathfrak{o}(\boldsymbol{a}).$$

It can be proved, by a non-constructive argument, that all the ordinals ω_a are contained within the second number class. Nevertheless there is an analogy between these ordinals ω_a and the first ordinals of the successive number classes, in which the notion of λ-definability corresponds, in general, to the notion of existence. In particular, Ω, the first ordinal of the third number class, can be described as the least ordinal of the second kind which is not the limit of an increasing sequence of ordinals of order type ω. Correspondingly, ω_1 is the least ordinal of the second kind which is not the limit of a λ-definable increasing sequence of ordinals of order type ω.

In fact, if we are willing to take the position that an ordinal less than ω_1 is not constructively calculated unless a formula representing it is constructively calculated, then we may regard an infinite sequence of ordinals (of order type ω) as constructive if some function f such that $f(n)$ is the Gödel representation of a formula representing the nth ordinal of the sequence is effectively calculable in the sense of Church, *An unsolvable problem of elementary number theory*, and hence conclude, by use of $\mathfrak{J}$ and of Kleene, *λ-definability and recursiveness*, (25), that ω_1 is not the limit of any constructive increasing sequence of ordinals of order type ω.