Introduction to Propositional Satisfiability

IA085: Satisfiability and Automated Reasoning

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FI MUNI, Spring 2024

Contents

Propositional satisfiability (SAT)

- $(A \vee \neg B) \wedge (\neg A \vee C)$
- is it satisfiable?

Satisfiability modulo theories (SMT)

- $x = 1 \;\wedge\; x = y + y \;\wedge\; y > 0$
- is it satisfiable over reals?
- is it satisfiable over integers?

Automated theorem proving (ATP)

- axioms: $\forall x\,(x + x = 0)$, $\forall x \forall y\,(x + y = y + x)$
- do they imply $\forall x \forall y\,((x + y) + (y + x) = 0)$?

Contents

For each of the problems (SAT/SMT/ATP)

- necessary definitions and theoretical results
- algorithms to solve the problem
- usage in practice and practical considerations

Organization of the Course

Schedule and Requirements

During semester

- lecture every week (except April 10, May 1, May 8)
- seminar every other week
- project (write your own small SAT solver) -- mandatory

Exam

- oral exam
- you will have access to the lecture slides

Project

Implement your own SAT solver

- you can use any reasonable programming language (C, C++, C#, Go, Java, Python, Rust, . . .)
- you are encouraged to work in pairs (but you do not have to)
- technical requirements are specified in the information system
- more advanced features $\rightarrow$ bonus points for the exam
- the scores will be evaluated periodically through the semester, you will see the ranking

Who am I?

- author of SMT solver Q3B for quantified formulas over bit-vector theory
- for 3 years post-doctoral researcher in Fondazione Bruno Kessler: research focused on SMT-based verification of software and SAT-based verification of hardware
- PhD thesis about satisfiability of quantified formulas over bit-vector theory
- author of several research papers about solving SMT and using it in practice
- co-organizer of SMT-COMP 2024

Propositional Logic

Propositional logic

Propositional logic deals with propositions, their relationships, and arguments based on them.

Does not deal objects and their properties, just with separate atomic claims.

``Martin has brown hair''
``Martin does not have hair''
No relationship as far as propositional logic is concerned.

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Implies *``Martin does have hair''* .

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No relationship as far as propositional logic is concerned.

``Martin has brown hair'' (A)
if ``Martin has brown hair`` then ``Martin does have hair'' ($A \rightarrow B$)
Implies *``Martin does have hair''* (B).

Syntax

Let $V = \{A, B, C, \ldots\}$ be a countable set of propositional variables. The set of propositional formulas is defined inductively as

- $\top$ and $\bot$ are propositional formulas,
- v is a propositional formula for each $v \in V$ (called propositional atom),
- if φ is a propositional formula, $\neg\varphi$ is a propositional formula,
- if φ and ψ are propositional formulas, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, and $\varphi \leftrightarrow \psi$ are propositional formulas.

Example

- $A \wedge B$
- $(A \vee B) \leftrightarrow \neg C$

Formulas of form v or $\neg v$ are called literals.

Semantics: Truth Assignments

$Atoms(\varphi)$ = the set of all atoms of formula φ.

(Total) truth assignment for formula φ

- assigns true ($\top$) or false ($\bot$) to each propositional variable in φ
- a function $\mu \colon V' \to \{\top, \bot\}$ where $Atoms(\varphi) \subseteq V'$
- can be written as a set of non-contradictory literals containing all variables of φ

Example

- formula $\varphi = A \vee B$,
- total assignment $\mu(A) = \top, \mu(B) = \bot$,
- written as $\mu = \{A, \neg B\}$.

Semantics: Satisfaction

Define when a truth assignment μ satisfies the formula φ, written $\mu \models \varphi$:

- $\mu \models \top$,
- $\mu \models v$ if $\mu(v) = \top$,
- $\mu \models \neg\psi$ if not $\mu \models \psi$,
- $\mu \models \psi_1 \wedge \psi_2$ if $\mu \models \psi_1$ and $\mu \models \psi_2$,
- $\mu \models \psi_1 \vee \psi_2$ if $\mu \models \psi_1$ or $\mu \models \psi_2$,
- $\mu \models \psi_1 \rightarrow \psi_2$ if not $\mu \models \psi_1$ or $\mu \models \psi_2$,
- $\mu \models \psi_1 \leftrightarrow \psi_2$ if $\mu \models \psi_1$ if and only if $\mu \models \psi_2$,

Semantics: Model

If $\mu \models \varphi$, we say that μ is a model of φ.

Example
$\{A, \neg B, C\}$ is a model of $A \wedge (B \leftrightarrow \neg C)$

An assignment μ is a partial model of φ if each extension of μ that is a truth assignment to φ is a model of φ.

Example
$\{A, B\}$ is a partial model of $(A \wedge B) \vee (A \wedge C)$

Propositional Entailment

Formula φ propositionally entails formula ψ (written $\varphi \models \psi$) if every μ that is a truth assignment for both φ and ψ satisfies

$$\text{if } \mu \models \varphi \text{ then also } \mu \models \psi$$

Example

- $A \models A \vee B$
- $(A \rightarrow B) \wedge A \models B$
- $(A \vee B) \wedge (\neg A \vee C) \models (B \vee C)$
- $A \not\models A \wedge B$

Formulas φ and ψ are propositionally equivalent (written $\varphi \equiv \psi$) if

$$\varphi \models \psi \quad \text{and} \quad \psi \models \varphi$$

Example

- $A \wedge (B \vee C) \;\equiv\; (A \wedge B) \vee (A \wedge C)$
- $A \wedge (A \vee B) \;\equiv\; A$
- $\neg(A \wedge B) \;\equiv\; \neg A \vee \neg B$

Negation Normal Form (NNF)

- negations are applied only to propositional atoms
- the formula does not contain implication ($\rightarrow$) and equivalence ($\leftrightarrow$)

Transformation to NNF

1. rewrite all $\varphi \leftrightarrow \psi$ to $(\varphi \rightarrow \psi) \land (\varphi \leftarrow \psi)$
2. rewrite all $\varphi \rightarrow \psi$ to $\neg\varphi \lor \psi$
3. apply De Morgan rules until fixed point
   - rewrite $\neg(\varphi \land \psi)$ to $(\neg\varphi) \lor (\neg\psi)$
   - rewrite $\neg(\varphi \lor \psi)$ to $(\neg\varphi) \land (\neg\psi)$

Conversion to NNF: Complexity

What is the complexity of conversion to NNF?

$$\varphi \leftrightarrow \psi \quad \rightsquigarrow \quad (\neg\varphi \vee \psi) \wedge (\varphi \vee \neg\psi)$$

Each equivalence doubles the size of the formula $\rightarrow$ translation can be exponential!

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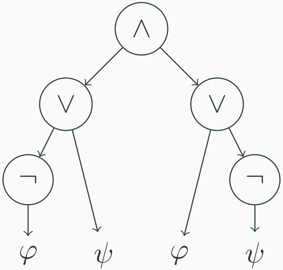
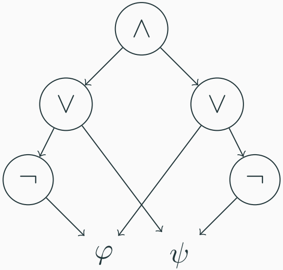
Or is it? It depends on the representation of the formulas.

$$(\neg\varphi \vee \psi) \wedge (\varphi \vee \neg\psi)$$

Tree



Directed acyclic graph (DAG)



In practice, we represent formulas as DAGs.

Conversion to NNF: Complexity

Theorem
When representing formulas as DAGs, the transformation to NNF is linear.

Proof idea.
The DAG contains two nodes for each subformula φ: one for φ, one for $\neg\varphi$. $\square$

Conversion to NNF: Complexity

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When representing formulas as DAG*s, the transformation to* NNF *is linear.*

Proof idea.
The DAG contains two nodes for each subformula φ: one for φ, one for $\neg\varphi$. $\square$

Proof details (bonus).
Recursively define function $NNF(\varphi) = (\varphi^+, \varphi^-)$. Given $NNF(\psi) = (\psi^+, \psi^-)$ and $NNF(\rho) = (\rho^+, \rho^-)$:

$$NNF(\psi \wedge \rho) = (\psi^+ \wedge \rho^+,\ \psi^- \vee \rho^-)$$

$$NNF(\neg\psi) = (\psi^-,\ \psi^+)$$

$$NNF(\psi \leftrightarrow \rho) = (\underbrace{(\psi^- \vee \rho^+) \wedge (\psi^+ \vee \rho^-)}_{\text{positive}},\ \underbrace{(\psi^+ \wedge \rho^-) \vee (\psi^- \wedge \rho^+)}_{\text{negative}}).$$

For more details, see *Property 1* in *Gabriele Masina, Giuseppe Spallitta, Roberto Sebastiani: On CNF Conversion for SAT Enumeration.* $\square$

Conjunctive Normal Form

Clause

- disjunction of literals
- $A \vee \neg B \vee C$
- written as $\{A, \neg B, C\}$ thanks to idempotence, commutativity, and associativity
- what is $\{\}$?

Formula in Conjunctive Normal Form (CNF)

- conjunction of clauses
- $(A \vee \neg B \vee C) \wedge (B \vee \neg C) \wedge C$
- written as $\{\{A, \neg B, C\}, \{B, \neg C\}, \{C\}\}$ thanks to idempotence, commutativity, and associativity
- what is $\{\}$?
- what is $\{\emptyset\}$?

Conjunctive Normal Form

- Easy to represent (`clause = list[int]`, `formula = list[clause]`).
- Easy to write algorithms, do not have to deal with the structure of the formula.
- Most of modern SAT solvers have input in CNF.

Conjunctive Normal Form

Transformation to CNF (naive)

1. transform to NNF
2. apply distributivity until fixed point
   - rewrite $\varphi \vee (\psi \wedge \rho)$ to $(\varphi \vee \psi) \wedge (\varphi \vee \rho)$
   - rewrite $(\psi \wedge \rho) \vee \varphi$ to $(\psi \vee \varphi) \wedge (\rho \vee \varphi)$

This is again exponential, try with $\bigvee_{1 \leq i \leq n} (A_i \wedge B_i)$.

Can we do better? What if the DAG representation is used? What if we use a different algorithm?

Conversion to CNF: Naive

Theorem

There exists an infinite family of formulas $\Phi = \{\varphi_i \mid i \in \mathbb{N}\}$ *such that for each equivalent family of formulas with* $\varphi_i^{CNF} \equiv \varphi_i$*, the size* $|\varphi_i^{CNF}|$ *grows exponentially with respect to* $|\varphi_i|$ *(even for* DAG *representation).*

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Proof.

Let $parity_i(A_1, A_2, \ldots, A_i) = A_1 \oplus A_2 \oplus \ldots \oplus A_i$. We can show that

- $parity_i$ can be defined by a formula φ_i of size $\mathcal{O}(i)$,
- each formula φ_i^{CNF} in CNF that defines $parity_i$ has 2^{i-1} clauses. □

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We cannot do better than exponential. ☹

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We cannot do better than exponential. ☹

Or can we? Yes, we can! Later today.

Disjunctive Normal Form

Cube

- conjunction of literals
- $A \wedge \neg B \wedge C$

Formula in Disjunctive Normal Form (DNF)

- disjunction of cubes
- $(A \wedge \neg B \wedge C) \vee (B \wedge \neg C) \vee C$

We will not be dealing with DNF often in this course.

Propositional Satisfiability (SAT)

Problem (SAT)
Given a propositional formula, decide whether it is satisfiable.

Problem (CNF-SAT)
Given a propositional formula in CNF, *decide whether it is satisfiable.*

Problem (3-SAT)
Given a propositional formula in CNF *with each clause of size 3, decide whether it is satisfiable.*

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Hardness of Propositional Satisfiability: Theory

Theorem
SAT, CNF-SAT, and 3-SAT are NP-complete.

Proof ideas.

- Whether an assignment is a model can be checked in polynomial time.
- A computation of Turing machine of polynomial length can be encoded by a CNF formula of polynomial size. □

There are no known polynomial algorithms for propositional satisfiability.

Hardness of Propositional Satisfiability: Practice

Modern SAT solvers can decide satisfiability of formulas with thousands of variables and millions of clauses thanks to

- clever algorithms (worst case exponential)
- clever data structures
- clever heuristics

Give it a try:

- MiniSAT (`http://minisat.se/`)
- CaDiCaL (`https://github.com/arminbiere/cadical`)
- Kissat (`https://github.com/arminbiere/kissat`)

Applications: Other Logical Problems

Other logical problems can be reduced[1] to satisfiability

Entailment

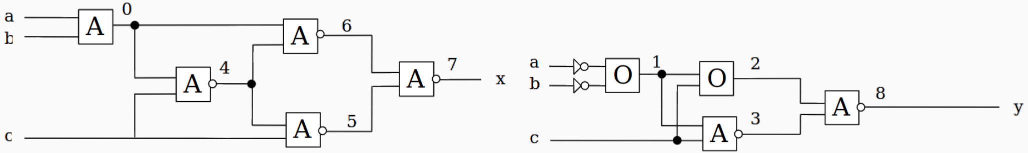
- $\varphi \models \psi \quad \Leftrightarrow \quad \varphi \land \neg\psi$ is not satisfiable

Validity

- a φ is valid if every total assignment for φ is its model
- φ is valid $\quad \Leftrightarrow \quad \neg\varphi$ is not satisfiable

[1] in the sense of Turing reductions

Applications: Hardware Design



[Example from: https://www21.in.tum.de/~lammich/2015_SS_Seminar_SAT/resources/Equivalence_Checking_11_30_08.pdf]

Are circuits C_1 and C_2 equivalent?

Is $\neg(\mathit{formula}(C_1) \leftrightarrow \mathit{formula}(C_2))$ UNSAT? (called a miter formula)

Works only for reasonably small circuits. For larger circuits (millions of gates), more involved techniques are necessary, e.g., SAT-sweeping.

Applications: Package Dependency

- package P has n versions:
  $x_1^P, x_2^P, \ldots, x_n^P$
- only one can be installed at a time:
  $\neg x_i^P \vee \neg x_j^P$ for all packages P and versions $i \neq j$
- packages have dependencies:
  $x_3^P \rightarrow (x_1^Q \vee x_2^Q) \wedge x_8^R$
- I have version 1 of package Q and want to install version 3 of package P:
  $x_3^P \wedge x_8^Q$
- what dependencies I need to install:
  Is the formula SAT? What is its model?

Used for example by package manager Cabal for Haskell.

Definition
A triple $(a, b, c) \in \mathbb{N}$ is called Pythagorean if $a^2 + b^2 = c^2$.

Question
Can every set of numbers $N = \{1, 2, \ldots, n\}$ be colored by two colors such that there is no monochromatic Pythagorean triple?

[2] `https://www.cs.utexas.edu/~marijn/ptn/`

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The answer is no ($n = 7825$) and was found by a SAT solver in 2016[2]. Previous lower bound was that $n = 7664$ can be colored.

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Applications: Open Problems in Mathematics

1. Define a formula F_i whose models are two-colorings of $\{1, 2, \ldots, i\}$ with no monochromatic Pythagorean triples.

$$F_i = \bigwedge_{(a,b,c) \text{ is a Pythagorean triple}} (x_a \vee x_b \vee x_c) \wedge (\neg x_a \vee \neg x_b \vee \neg x_c)$$

2. F_{7824}: 6492 variables and 18930 clauses; F_{7825}: 6494 variables and 18944 clauses.
3. Preprocessing: reduce this to 3740 variables and 14652 clauses; and 3745 variables and 14672 clauses.
4. Use parallel SAT solver and tweak some of its heuristics.
5. Use a parallel machine with 800 cores for 2 days.
6. Find that F_{7824} is satisfiable and F_{7825} is unsatisfiable.
7. Get a largest unsatisfiability proof ever (200 terabytes).

Thinking with Clauses

Clauses = Implications

Important view during this course: clauses = implications.

$\{A, B\}$ (i.e., $A \vee B$)

- $\neg A \rightarrow B$
- $\neg B \rightarrow A$

$\{A, B, C\}$ (i.e., $A \vee B \vee C$)

- $(\neg A \wedge \neg B) \rightarrow C$
- $(\neg A \wedge \neg C) \rightarrow B$
- $\ldots$

2-SAT

2-CNF = formula in CNF with clauses of size 2

2-SAT = decide satisfiability of formula in 2-CNF

Example

Is the following 2-CNF formula satisfiable?

$$\{A, B\}, \quad \{\neg B, C\}, \quad \{\neg C, A\}$$
$$\{\neg A, \neg B\}, \quad \{B, \neg A\}, \quad \{C, \neg D\}$$

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Theorem

2-SAT can be solved in linear time.

Theorem
*2-*SAT *can be solved in linear time.*

Theorem
2-SAT can be solved in linear time.

Proof.
Let φ be in 2-CNF. Construct a graph $G = (V, E)$ with

- $V = \{v \mid v \in Atoms(\varphi)\} \cup \{\neg v \mid v \in Atoms(\varphi)\}$
- $E = \{(\neg a, b) \mid \{a, b\} \in \varphi\}$

φ is satisfiable $\quad \Leftrightarrow \quad$ G has no cycle that contains both v and $\neg v$ for some v $\qquad \square$

Encoding Graph Coloring

Given an undirected graph $G = (V, E)$, can it be colored by three colors (red, green, blue) so that no edge has endpoints of the same color?

Encoding

- variables v_r, v_g, v_b for each $v \in V$

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- coloring constraint $u_c \to \neg v_c$ for each edge $\{u, v\} \in E$ and each color $c \in \{r, g, b\} \equiv$ clause $\{\neg u_c, \neg v_c\}$

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- coloring constraint $u_c \rightarrow \neg v_c$ for each edge $\{u, v\} \in E$ and each color $c \in \{r, g, b\} \equiv$ clause $\{\neg u_c, \neg v_c\}$
- models of the formula $\simeq$ valid colorings

Conversion to equivalent CNF can be exponential, but do we really need equivalence?

Definition
The formulas φ and ψ are equisatisfiable if both are satisfiable or both unsatisfiable.

Theorem
For each formula φ there exists an equisatisfiable formula φ^{CNF} with $\mathcal{O}(|\varphi|)$ clauses of size at most three.

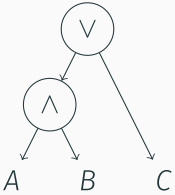
Proof.
Tseitin encoding. □

Conversion to CNF: Tseitin encoding by example

$\varphi = (A \wedge B) \vee C$

Conversion to CNF: Tseitin encoding by example

$\varphi = (A \land B) \lor C$



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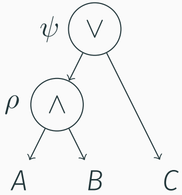
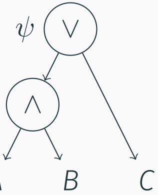


Conversion to CNF: Tseitin encoding by example

$\varphi = (A \wedge B) \vee C$

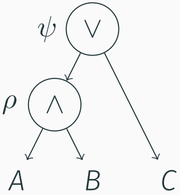


$(A_\rho \leftrightarrow (A \wedge B)) \wedge$

$(A_\psi \leftrightarrow (A_\rho \vee C)) \wedge$

A_ψ

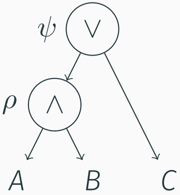
$\varphi = (A \land B) \lor C$



$$
\begin{array}{lcl}
(A_\rho \leftrightarrow (A \land B)) \land & & (A_\rho \rightarrow (A \land B)) \land \\
 & & (A_\rho \leftarrow (A \land B)) \land \\
(A_\psi \leftrightarrow (A_\rho \lor C)) \land & \equiv & (A_\psi \rightarrow (A_\rho \lor C)) \land \\
 & & (A_\psi \leftarrow (A_\rho \lor C)) \land \\
A_\psi & & A_\psi
\end{array}
$$

Conversion to CNF: Tseitin encoding by example

$\varphi = (A \wedge B) \vee C$



$$
\begin{array}{lclcl}
(A_\rho \leftrightarrow (A \wedge B)) \wedge & & (A_\rho \rightarrow (A \wedge B)) \wedge & & \{\neg A_\rho, A\}, \{\neg A_\rho, B\}, \\
 & & (A_\rho \leftarrow (A \wedge B)) \wedge & & \{\neg A, \neg B, A_\rho\}, \\
(A_\psi \leftrightarrow (A_\rho \vee C)) \wedge & \equiv & (A_\psi \rightarrow (A_\rho \vee C)) \wedge & \equiv & \{\neg A_\psi, A_\rho, C\}, \\
 & & (A_\psi \leftarrow (A_\rho \vee C)) \wedge & & \{\neg A_\rho, A_\psi\}, \{\neg C, A_\psi\}, \\
A_\psi & & A_\psi & & \{A_\psi\}
\end{array}
$$

Conversion to CNF: Tseitin encoding

1. Create a new Tseitin variable A_ψ for each subformula of φ.
2. Add unit clause $\{A_\varphi\}$.
3. Define semantics of the new Tseitin variables A_ψ:

ψ	definition of A_ψ	added clauses
$\rho_1 \vee \rho_2$	$A_\psi \rightarrow (A_{\rho_1} \vee A_{\rho_2})$ $A_\psi \leftarrow (A_{\rho_1} \vee A_{\rho_2})$	$\{\neg A_\psi, A_{\rho_1}, A_{\rho_2}\}$ $\{\neg A_{\rho_1}, A_\psi\}, \{\neg A_{\rho_2}, A_\psi\}$
$\rho_1 \wedge \rho_2$	$A_\psi \rightarrow (A_{\rho_1} \wedge A_{\rho_2})$ $A_\psi \leftarrow (A_{\rho_1} \wedge A_{\rho_2})$	$\{\neg A_\psi, A_{\rho_1}\}, \{\neg A_\psi, A_{\rho_2}\}$ $\{\neg A_{\rho_1}, \neg A_{\rho_2}, A_\psi\}$
$\neg \rho$	$A_\psi \rightarrow \neg A_\rho$ $A_\psi \leftarrow \neg A_\rho$	$\{\neg A_\psi, \neg A_\rho\}$ $\{A_\rho, A_\psi\}$
$\rho_1 \leftrightarrow \rho_2$	$A_\psi \rightarrow (A_{\rho_1} \leftrightarrow A_{\rho_2})$ $A_\psi \leftarrow (A_{\rho_1} \leftrightarrow A_{\rho_2})$	$\{\neg A_\psi, \neg A_{\rho_1}, A_{\rho_2}\}, \{\neg A_\psi, A_{\rho_1}, \neg A_{\rho_2}\}$ $\{\neg A_{\rho_1}, \neg A_{\rho_2}, A_\psi\}, \{A_{\rho_1}, A_{\rho_2}, A_\psi\}$

Tseitin encoding

- often used in practice
- also works for DAG representation of formulas $\rightarrow$ one Tseitin variable for each node in the DAG
- transforming to increase shared subexpression helps $(B \wedge A) \leadsto (A \wedge B)$
- additional preprocessing helps: $(A \vee (B \vee C)) \leadsto (A \vee B \vee C)$ and then encode $A_\varphi \leftrightarrow (A \vee B \vee C)$ as one Tseitin variable and four implications
- some of the clauses are not needed (Plaisted-Greenbaum)

Next time

Classical SAT algorithms

- propositional resolution
- Davis-Putnam-Logemann-Loveland algorithm (DPLL)