Introduction to First-Order Satisfiability

IA085: Satisfiability and Automated Reasoning

Martin Jonáš

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Where are we?

Satisfiability modulo theories (SMT)

- $x = 1 \land x = y + y \land y > 0$
- is it satisfiable over reals?
- is it satisfiable over integers?
- is it satisfiable over integers represented by 8 bits?
- is it satisfiable over floating point numbers represented by 32 bits?

For next four lectures, we will be dealing mostly with quantifier-free formulas.

Applications

SMT solvers are widely used in practice

- planning
- scheduling
- verification of hardware
- compiler optimizations
- verification of software
- ...

Applications

```
int x = read();
int y = read();
int z = read();
if (x > 10 && y != 0)
{
  print(z / (x + y));
}
```

Contains division by zero precisely if the formula

$$x > 10 \quad \wedge \quad \neg(y = 0) \quad \wedge \quad x + y = 0$$

is satisfiable.

First-order logic

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- $\exists s.\, \mathrm{Human}(s) \;\wedge\; \mathrm{Mortal}(s).$
- $\forall s.\, \mathrm{Human}(s) \;\rightarrow\; \mathrm{Mortal}(s).$
- $\exists x \exists y.\, x < 5 \;\wedge\; y < 3 \;\wedge\; 2 \cdot (x + y) > 20.$

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In addition to logical symbols, first-order formulas contain variables, constant symbols, function symbols, and predicate symbols.

First-Order Logic – Syntax

Suppose we have

- a set $\Sigma^F = \{f, g, \ldots\}$ of function symbols and
- a set $\Sigma^P = \{R, S, \ldots\}$ of predicate symbols.

Each function symbol f and predicate symbol P has its arity $\mathrm{ar}(f)$ and $\mathrm{ar}(P)$.
Function symbols of arity 0 are called constants.

The set $\Sigma = \Sigma^F \cup \Sigma^P$ is called a signature.

Example

- $\Sigma^F = \{+, -, 0, 1\}$
- $\Sigma^P = \{=, \leq\}$

(Σ-)Term

1. a variable – $x, y, z, \ldots$
2. a function symbol applied to $\mathrm{ar}(f)$ terms – $f(x)$, $g(f(x), y)$, ...

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(Σ)-Formula

1. a Boolean combination of literals – $(R(x) \vee \neg R(y)) \wedge S(f(x), y), \ldots$
2. a quantifier applied to a formula – $\forall x \left(R(x)\right), \ldots$.

First-Order Logic – Syntactic Conventions and Terminology

Notation

- instead of $+(r, s)$ write $r + s$ (also for other infix function symbols)
- instead of $\leq (r, s)$ write $r \leq s$ (also for other infix predicate symbols)
- instead of 1() write 1 (also for other constants)
- instead of $\forall x \forall y (\varphi \wedge \psi)$ write $\forall x \forall y.\, \varphi \wedge \psi$

Terminology

- an occurrence of a variable is free if it is not bound by a quantifier
- a formula without free occurrences of variables is closed or a sentence

First-Order Logic – Semantics

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Meaning of these three things is given by a Σ-structure.

Σ-structure $\mathcal{A}$

- determines the set of objects and behavior of functions/predicates
- a pair of
  1. a non-empty set A called the universe,
  2. a map $(\_)^{\mathcal{A}}$ that
     - to each $f \in \Sigma^F$ assigns a function $f^{\mathcal{A}} \colon A^{\mathrm{ar}(f)} \to A$,
     - to each $R \in \Sigma^P \setminus \{=\}$ assigns a relation $R^{\mathcal{A}} \subseteq A^{\mathrm{ar}(R)}$,
     - we suppose that $=^A$ is the identity relation.

$\mathcal{A} = \big(A, (\_)^{\mathcal{A}}\big)$ where

$$
\begin{aligned}
A &= \mathbb{Z} \\
+^{\mathcal{A}}(x, y) &= x + y \\
<^{\mathcal{A}} &= \{(x, y) \mid x < y\} \\
1^{\mathcal{A}} &= 1
\end{aligned}
$$

First-Order Logic – Structure Examples

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$\mathcal{B} = \big(B, (\_)^{\mathcal{B}}\big)$ where

$$\begin{aligned} B &= \{\circ, \bullet\} \\ +^{\mathcal{B}}(x, y) &= y \\ <^{\mathcal{B}} &= \{(\circ, \circ), (\bullet, \circ)\} \\ 1^{\mathcal{B}} &= \bullet \end{aligned}$$

First-Order Logic – Interpretation

The formulas can also contain free variables.

Valuation for a Σ-structure $\mathcal{A} = (A, (\_)^{\mathcal{A}})$

- determines the values of the variables
- a map $\mu \colon \mathit{Vars} \to A$

Σ-interpretation

- a pair $(\mathcal{A}, \mu)$ of a Σ-structure and a valuation

Given an interpretation $\mathcal{I} = (\mathcal{A}, \mu)$, we can evaluate

- each term t to a value $[\![t]\!]^{\mathcal{I}} \in A$
- each formula φ to a value $[\![\varphi]\!]^{\mathcal{I}} \in \{\top, \bot\}$

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Interpretation $\mathcal{I}$ satisfies formula φ

- if $[\![\varphi]\!]^{\mathcal{I}} = \top$
- written $\mathcal{I} \models \varphi$

Entailment and Validity

Formula φ **entails** formula ψ

- if every interpretation that satisfies φ also satisfies ψ
- written $\varphi \models \psi$
- example: $f(x) = y \land x = z \;\models\; f(z) = y$
- negative example: $x < y \;\not\models\; x + 1 < y + 1$

Formula φ is **valid**

- if every interpretation satisfies φ
- written $\models \varphi$
- example: $\models P(f(x)) \lor \neg P(f(x))$
- negative example: $\not\models x + 0 = x$

First-Order Satisfiability

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Formula φ is satisfiable if there is a Σ-interpretation $(\mathcal{A}, \mu)$ such that $(\mathcal{A}, \mu) \models \varphi$.

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Solution

Pick a subset of Σ-structures in which we are interested.

This gives rise to the Satisfiability Modulo Theories

Satisfiability Modulo Theories (SMT)

Definition
A (Σ-)theory is a set of Σ-structures.

Definition
A formula φ is satisfiable modulo theory T if there is a Σ-interpretation $(\mathcal{A}, \mu)$ with $\mathcal{A} \in T$ such that $(\mathcal{A}, \mu) \models \varphi$.

Satisfiability Modulo Theories – Example

Consider the structure $\mathcal{Z}$ with the universe $\mathbb{Z}$ and the standard interpretation of operations $+$, $<$, and 1.

The formula $(x < y) \ \wedge \ (y + 1 < x)$ is unsatisfiable modulo theory $T = \{\mathcal{Z}\}$.

The formula $(x < y) \ \wedge \ (y < x + 2)$ is satisfiable modulo theory $T = \{\mathcal{Z}\}$.

Linear Integer Arithmetic (LIA)

- $\Sigma = \{0, 1, +, -, =, \leq\}$
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T-models

$\mathcal{I} = (\mathcal{A}, \mu)$ is T-model of φ

- if $\mathcal{A} \in T$ and $[\![\varphi]\!]^{\mathcal{I}} = \top$
- written $\mathcal{I} \models_T \varphi$

Entailment and Validity

φ **T-entails** ψ

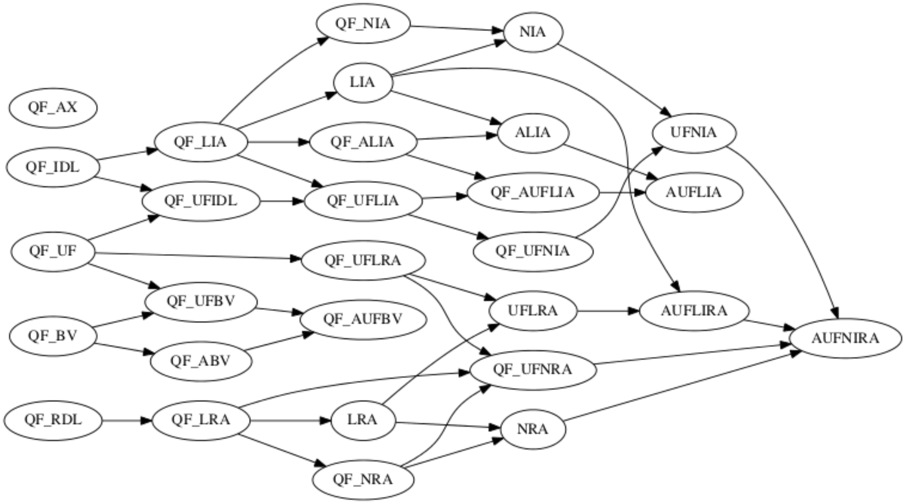
- if every T-model of φ is also a T-model of ψ
- written $\varphi \models_T \psi$
- example: $x < y \models_{T_{\mathrm{LIA}}} x + 1 < y + 1$

φ is **T-valid**

- if every $\mathcal{I} = (\mathcal{A}, \mu)$ with $\mathcal{A} \in T$ is a T-model of φ
- equivalently $\top \models_T \varphi$
- written $\models_T \varphi$
- example: $\models_{T_{\mathrm{LIA}}} x + 0 = x$

Theories of interest

Logics in SMT-LIB



[https://smtlib.cs.uiowa.edu/logics.shtml]

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- satisfiability of quantifier-free formulas is NP-complete
- satisfiability of conjunctions of literals in P (Khachiyan, 1979)

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- satisfiability of conjunctions of quantifier-free formulas is undecidable (Matiyasevich, 1971)

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More theories

- theory of strings ($s_1 \mathbin{+\!\!+}\ s_2$, $\mathrm{len}(s)$, $\mathrm{substr}(s, \mathit{from}, \mathit{to})$, $\mathrm{contains}(s_1, s_2)$, $\ldots$)

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Standard view of theories

Definition
A (Σ-)theory is a set of closed Σ-formulas.

Definition
A formula φ is satisfiable modulo theory T if there is a Σ-interpretation $\mathcal{I}$ such that

- $\mathcal{I} \models \varphi$ and
- $\mathcal{I} \models \psi$ for all $\psi \in T$

Linear Natural Arithmetic

- $\Sigma = \{0, 1, +, =, \leq\}$

SMT definition

- $T = \{(\mathbb{N}, (\_)^{\mathbb{N}})\}$, where $(\_)^{\mathbb{N}}$ is the obvious standard interpretation

Standard definition (Presburger axioms)

$$
\begin{aligned}
T = \{&\forall x.\ \neg(0 = x + 1),\\
&\forall x \forall y.\ x + 1 = y + 1 \rightarrow x = y,\\
&\forall x.\ x + 0 = x,\\
&\forall x \forall y.\ x + (y + 1) = (x + y) + 1\} \cup\\
&\{(P(0) \wedge \forall x (P(x) \rightarrow P(x + 1))) \rightarrow \forall y P(y) \mid P \text{ is a formula with free variable } x\}
\end{aligned}
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Two views of theories

More examples

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- The axioms of theory of arrays (McCarthy):

$$
\begin{aligned}
T_A = \{\, & \forall a, i, j.\, (i = j \;\rightarrow\; \mathrm{read}(a, i) = \mathrm{read}(a, j)), \\
& \forall a, v, i, j.\, (i = j \;\rightarrow\; \mathrm{read}(\mathrm{write}(a, i, v), j) = v), \\
& \forall a, v, i, j.\, (i \neq j \;\rightarrow\; \mathrm{read}(\mathrm{write}(a, i, v), j) = \mathrm{read}(a, j))\}
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Next time

- algorithms solving SMT
- CDCL(T) algorithm