IA169 Model Checking

Reachability in Pushdown Systems

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Motivation

Pushdown systems can be used to precisely model sequential programs with procedure calls, unbounded recursion, and both local and global variables with finite domains.

Agenda and sources

agenda

- pushdown systems
- representation of sets of configurations
- computing all predecessors
- bonus: state-based LTL model checking

sources

- J. Esparza, D. Hansel, P. Rossmanith, and S. Schwoon: *Efficient algorithms for model checking pushdown systems*, CAV 2000, LNCS 1855, Springer, 2000.
- S. Schwoon: *Model-Checking Pushdown Systems*, PhD thesis, TUM, 2002.

Pushdown systems

Pushdown systems

Definition (pushdown system)

A pushdown system is a triple $\mathcal{P} = (P, \Gamma, \Delta)$, where

- P is a finite set of control locations,
- Γ is a finite stack alphabet,
- $\Delta \subseteq (P \times \Gamma) \times (P \times \Gamma^*)$ is a finite set of transition rules.

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- we write $\langle q, \gamma \rangle \hookrightarrow \langle q', w \rangle$ instead of $((q, \gamma), (q', w)) \in \Delta$
- we do not consider any input alphabet as we do not use pushdown systems to represent languages

Definitions

- a configuration of $\mathcal{P}$ is a pair $\langle p, w \rangle \in P \times \Gamma^*$, where w is a stack content (the topmost symbol is on the left)
- the set of all configurations is denoted by $\mathcal{C}$
- an immediate successor relation on configurations is defined in standard way
- reachability relation $\Rightarrow \subseteq \mathcal{C} \times \mathcal{C}$ is the reflexive and transitive closure of the immediate successor relation
- $\overset{+}{\Rightarrow} \subseteq \mathcal{C} \times \mathcal{C}$ is the transitive closure of the immediate successor relation
- given a set $C \subseteq \mathcal{C}$ of configurations, we define the set of their predecessors as

$$pre^*(C) = \{c \in \mathcal{C} \mid \exists c' \in C \,.\, c \Rightarrow c'\}$$

Representation of sets of configurations

$\mathcal{P}$-automata

$\mathcal{P}$-automata

- are finite automata used to represent sets of configurations
- use Γ as an alphabet
- have one initial state for every control location of the pushdown (we use P as the set of initial states)

Definition ($\mathcal{P}$-automaton)

Given a pushdown system $\mathcal{P} = (P, \Gamma, \Delta)$, a $\mathcal{P}$-automaton (or simply automaton) is a tuple $\mathcal{A} = (Q, \Gamma, \delta, P, F)$ where

- Q is a finite set of states such that $P \subseteq Q$,
- $\delta \subseteq Q \times \Gamma \times Q$ is a set of transitions,
- $F \subseteq Q$ is a set of final states.

More definitions

- a (reflexive and transitive) transition relation $\rightarrow \subseteq Q \times \Gamma^* \times Q$ is defined in a standard way
- $\mathcal{P}$-automaton $\mathcal{A}$ represents the set of configurations

$$Conf(\mathcal{A}) = \{\langle p, w\rangle \mid \exists q \in F \,.\, p \xrightarrow{w} q\}$$

- a set of configurations of $\mathcal{P}$ is called regular if it is recognized by some $\mathcal{P}$-automaton

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notation convention

- $p, p', p'', \ldots$ denote initial states of an automaton (i.e. elements of P)
- $s, s', s'', \ldots$ denote non-initial states
- $q, q', q'', \ldots$ denote arbitrary states (initial or not)

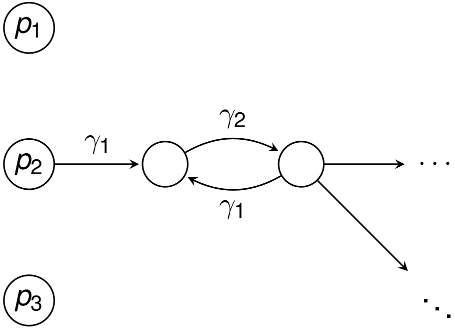
Computing all predecessors

Statements

1. Given a pushdown system $\mathcal{P}$ and a regular set of configurations C, the set $pre^*(C)$ is again regular.
2. If C is defined by a $\mathcal{P}$-automaton $\mathcal{A}$, then the automaton $\mathcal{A}_{pre^*}$ representing $pre^*(C)$ is effectively constructible.

Intuition

$$\langle p_1, \gamma_0 \rangle \hookrightarrow \langle p_2, \gamma_1 \gamma_2 \rangle$$
$$\langle p_3, \gamma_3 \rangle \hookrightarrow \langle p_1, \gamma_0 \gamma_1 \rangle$$



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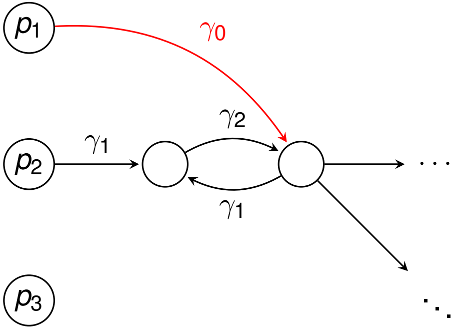

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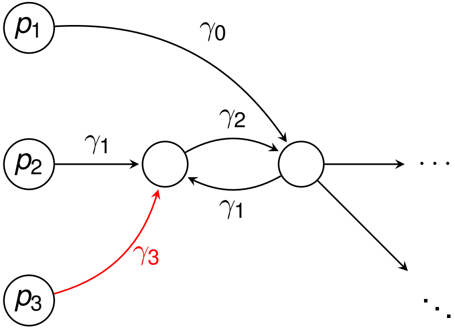


Idea

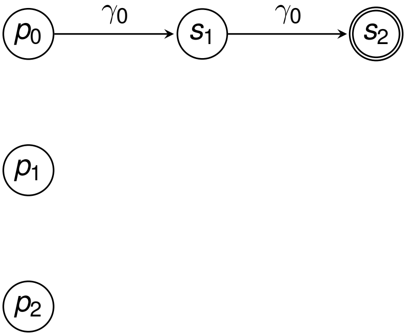
Let $\mathcal{P}$ be a pushdown system and $\mathcal{A}$ be a $\mathcal{P}$-automaton. We assume (w.l.o.g.) that $\mathcal{A}$ has no transition leading to an initial state. The automaton $\mathcal{A}_{pre^*}$ is obtained from $\mathcal{A}$ by addition of new transitions according to the following rule:

Saturation rule

If $\langle p, \gamma \rangle \hookrightarrow \langle p', w \rangle$ and $p' \xrightarrow{w} q$ in the current automaton,
add a transition (p, γ, q).

- we apply this rule repeatedly until we reach a fixpoint
- a fixpoint exists as the number of possible new transitions is finite
- the resulting $\mathcal{P}$-automaton is $\mathcal{A}_{pre^*}$

Example



transition rules of $\mathcal{P}$:

$$\langle p_0, \gamma_0 \rangle \hookrightarrow \langle p_1, \gamma_1 \gamma_0 \rangle \qquad \langle p_2, \gamma_2 \rangle \hookrightarrow \langle p_0, \gamma_1 \rangle$$
$$\langle p_1, \gamma_1 \rangle \hookrightarrow \langle p_2, \gamma_2 \gamma_0 \rangle \qquad \langle p_0, \gamma_1 \rangle \hookrightarrow \langle p_0, \varepsilon \rangle$$

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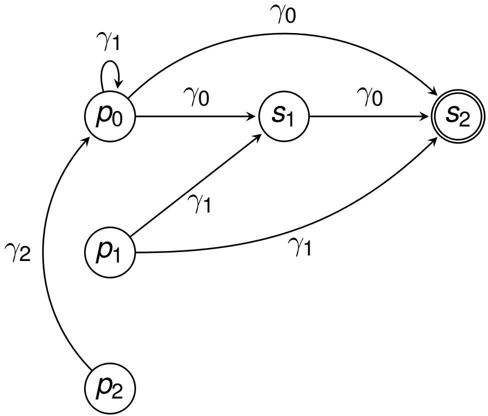


transition rules of $\mathcal{P}$:

$$\langle p_0, \gamma_0 \rangle \hookrightarrow \langle p_1, \gamma_1\gamma_0 \rangle \qquad \langle p_2, \gamma_2 \rangle \hookrightarrow \langle p_0, \gamma_1 \rangle$$
$$\langle p_1, \gamma_1 \rangle \hookrightarrow \langle p_2, \gamma_2\gamma_0 \rangle \qquad \langle p_0, \gamma_1 \rangle \hookrightarrow \langle p_0, \varepsilon \rangle$$

Normal form

Definition (normal form)

A pushdown system is in normal form if every rule $\langle p, \gamma \rangle \hookrightarrow \langle p', w \rangle$ satisfies $|w| \leq 2$.

- any pushdown system can be transformed into normal form with only linear size increase

Algorithm: notes

We give an algorithm that, for a given $\mathcal{A}$, computes transitions of $\mathcal{A}_{pre^*}$. The rest of the automaton $\mathcal{A}_{pre^*}$ is identical to $\mathcal{A}$.

The algorithm uses sets rel and trans containing the transitions that are known to belong to $\mathcal{A}_{pre^*}$:

- rel contains transitions that have already been examined
- no transition is examined more than once
- when we have a rule $\langle p, \gamma \rangle \hookrightarrow \langle p', \gamma' \gamma'' \rangle$ and transitions $t_1 = (p', \gamma', q')$ and $t_2 = (q', \gamma'', q'')$ (where q, q' are arbitrary states), we have to add transition (p, γ, q'')
- we do it in such a way that whenever we examine t_1, we check if there is a corresponding $t_2 \in$ rel and we add an extra rule $\langle p, \gamma \rangle \hookrightarrow \langle q', \gamma'' \rangle$ to a set of such extra rules Δ'
- the extra rule guarantees that if a suitable t_2 will be examined in the future, (p, γ, q'') will be added.

input : a pushdown system $\mathcal{P} = (P, \Gamma, \Delta)$ in normal form and
a $\mathcal{P}$-automaton $\mathcal{A}=(Q, \Gamma, \delta, P, F)$ without transitions into P
output: the set of transitions of $\mathcal{A}_{pre^*}$

rel $\leftarrow \emptyset$; trans $\leftarrow \delta$; $\Delta' \leftarrow \emptyset$
forall $\langle p, \gamma\rangle \hookrightarrow \langle p', \varepsilon\rangle \in \Delta$ **do** trans $\leftarrow$ trans $\cup \{(p, \gamma, p')\}$
while trans $\neq \emptyset$ **do**
    pop $t = (q, \gamma, q')$ from trans
    if $t \notin$ rel **then**
        rel $\leftarrow$ rel $\cup \{t\}$
        forall $\langle p_1, \gamma_1\rangle \hookrightarrow \langle q, \gamma\rangle \in (\Delta \cup \Delta')$ **do**
            trans $\leftarrow$ trans $\cup \{(p_1, \gamma_1, q')\}$
        forall $\langle p_1, \gamma_1\rangle \hookrightarrow \langle q, \gamma\gamma_2\rangle \in \Delta$ **do**
            $\Delta' \leftarrow \Delta' \cup \{\langle p_1, \gamma_1\rangle \hookrightarrow \langle q', \gamma_2\rangle\}$
            forall $(q', \gamma_2, q'') \in$ rel **do**
                trans $\leftarrow$ trans $\cup \{(p_1, \gamma_1, q'')\}$
return rel

Theorem

Theorem

Let $\mathcal{P} = (P, \Gamma, \Delta)$ be a pushdown system and $\mathcal{A} = (Q, \Gamma, \delta, P, F)$ be a $\mathcal{P}$-automaton. There exists an automaton $\mathcal{A}_{pre^*}$ recognizing $pre^*(Conf(\mathcal{A}))$. Moreover, $\mathcal{A}_{pre^*}$ can be constructed in $\mathcal{O}(|Q|^2 \cdot |\Delta|)$ time and $\mathcal{O}(|Q| \cdot |\Delta| + |\delta|)$ space.

Proof.

- We can assume that every transition is added to trans at most once. This can be done (without asymptotic loss of time) by storing all transitions which are ever added to trans in an additional hash table.
- Further, we assume that there is at least one rule in Δ for every $\gamma \in \Gamma$ (transitions of $\mathcal{A}$ under some γ not satisfying this assumption can be moved directly to rel).
- The number of transitions in δ as well as the number of iterations of the while-loop is bounded by $|Q|^2 \cdot |\Delta|$.

Proof: time complexity

Proof (Cont.)

- Line 10 is executed for each combination of a rule $\langle p_1, \gamma_1 \rangle \hookrightarrow \langle q, \gamma\gamma_2 \rangle$ and a transition $(q, \gamma, q') \in$ trans, i.e. at most $|Q| \cdot |\Delta|$ times.
- Hence, $|\Delta'| \leq |Q| \cdot |\Delta|$.
- For the loop starting at line 11, q' and γ_2 are fixed. Thus, line 12 is executed at most $|Q|^2 \cdot |\Delta|$ times.
- Line 8 is executed for each combination of a rule $\langle p_1, \gamma_1 \rangle \hookrightarrow \langle q, \gamma \rangle \in (\Delta \cup \Delta')$ and a transition $(q, \gamma, q') \in$ trans. As $|\Delta'| \leq |Q| \cdot |\Delta|$, line 8 is executed at most $\mathcal{O}(|Q|^2 \cdot |\Delta|)$ times.

As a conclusion, the algorithm takes $\mathcal{O}(|Q|^2 \cdot |\Delta|)$ time.

Proof: space complexity

Proof (Cont.)

Memory is needed for storing rel, trans, and Δ'.

- The size of Δ' is in $\mathcal{O}(|Q| \cdot |\Delta|)$.
- Line 1 adds $|\delta|$ transitions to trans.
- Line 2 adds at most $|\Delta|$ transitions to trans.
- In lines 8 and 12, p_1 and γ_1 are given by the head of a rule in Δ (note that every rule in Δ' have the same head as some rule in Δ). Hence, lines 8 and 12 add at most $|Q| \cdot |\Delta|$ different transitions.

We directly get that the algorithm needs $\mathcal{O}(|Q| \cdot |\Delta| + |\delta|)$ space. As this is also the size of the result rel, the algorithm is optimal with respect to the memory usage. □

Notes

- the algorithm can be used to verify safety property: given an automaton $\mathcal{A}$ representing error configurations, we can compute $\mathcal{A}_{pre^*}$, i.e. the set of all configurations from which an error configuration is reachable
- there is a similar algorithm computing, for a given regular set of configurations C, the set of all successors

$$post^*(C) = \{c' \in \mathcal{C} \mid \exists c \in C \,.\, c \Rightarrow c'\}$$

Theorem

Let $\mathcal{P} = (P, \Gamma, \Delta)$ be a pushdown system and $\mathcal{A} = (Q, \Gamma, \delta, P, F)$ be a $\mathcal{P}$-automaton. There exists an automaton $\mathcal{A}_{post^*}$ recognizing $post^*(Conf(\mathcal{A}))$. Moreover, $\mathcal{A}_{post^*}$ can be constructed in $\mathcal{O}(|P| \cdot |\Delta| \cdot (|Q| + |\Delta|) + |P| \cdot |\delta|)$ time and space.

Bonus: State-based LTL model checking

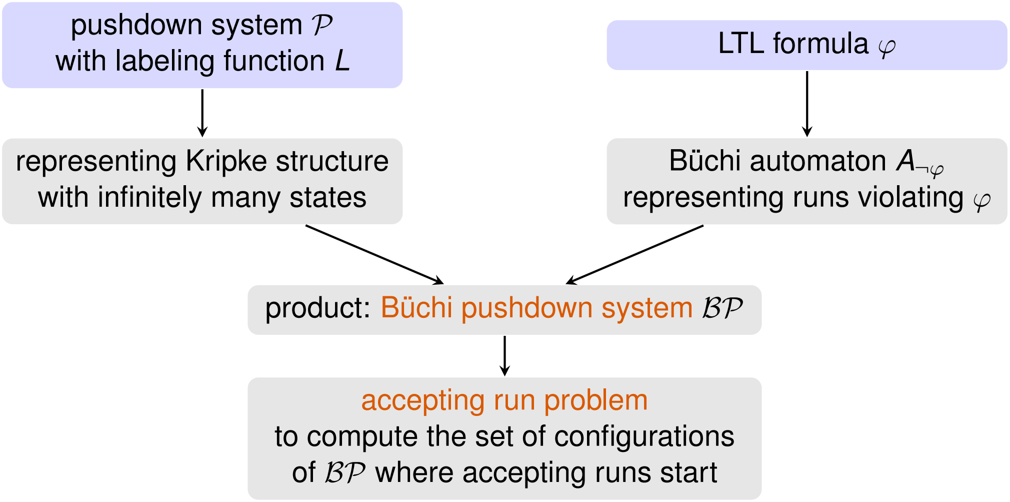
The problem

The global state-based LTL model checking problem for pushdown systems

Compute the set of all configurations of a given pushdown system $\mathcal{P}$ that violate a given LTL formula φ (where a configuration c violates φ if there is a path starting from c and not satisfying φ).

Extending pushdown systems

- state-based $\implies$ validity of atomic propositions
- labeling function $L : (P \times \Gamma) \rightarrow 2^{AP}$ assigns valid atomic propositions to every pair (p, γ) of a control location p and a topmost stack symbol γ
- pushdown system $\mathcal{P}$ and L define Kripke structure
  - states = configurations of $\mathcal{P}$
  - transition relation = immediate successor relation
  - no initial states (global model checking)
  - labeling function is an extension of L: $L(\langle p, \gamma w \rangle) = L(p, \gamma)$



pushdown system $\mathcal{P}$
with labeling function L
representing Kripke structure
with infinitely many states
LTL formula φ
Büchi automaton $A_{\neg\varphi}$
representing runs violating φ
product: Büchi pushdown system $\mathcal{BP}$
accepting run problem
to compute the set of configurations
of $\mathcal{BP}$ where accepting runs start

Büchi pushdown system

Büchi pushdown system

- pushdown system with a set of accepting control locations
- an accepting run is a path passing through some accepting control location infinitely often

Product

Product of

- a pushdown system $\mathcal{P} = (P, \Gamma, \Delta)$ with a labeling function L and
- a Büchi automaton $\mathcal{A}_{\neg\varphi} = (Q, 2^{AP(\varphi)}, \delta, Q_0, F)$

is a Büchi pushdown system $\mathcal{BP} = ((P \times Q), \Gamma, \Delta', G)$, where

$$\langle (p, q), \gamma \rangle \hookrightarrow \langle (p', q'), w \rangle \in \Delta' \quad \text{iff} \quad \begin{array}{l} \langle p, \gamma \rangle \hookrightarrow \langle p', w \rangle \in \Delta \text{ and} \\ (q, L(p, \gamma) \cap AP(\varphi), q') \in \delta \end{array}$$

and $G = P \times F$ it the set of accepting control locations.

Clearly, a configuration $\langle p, w \rangle$ of $\mathcal{P}$ violates φ iff $\mathcal{BP}$ has an accepting run starting from $\langle (p, q_0), w \rangle$ for some $q_0 \in Q_0$.

Accepting run problem

The original model checking problem reduces to the following:

The accepting run problem

Compute the set $\mathcal{C}_a$ of configurations c of $\mathcal{BP}$ such that $\mathcal{BP}$ has an accepting run starting from c.

Repeating heads

$\Rightarrow$ denotes the (reflexive and transitive) reachability relation

$\stackrel{+}{\Rightarrow}$ denotes the (transitive) reachability relation

Repeating heads

$\Rightarrow$ denotes the (reflexive and transitive) reachability relation
$\stackrel{+}{\Rightarrow}$ denotes the (transitive) reachability relation

We define the relation $\stackrel{r}{\Rightarrow}$ on configurations of $\mathcal{BP}$ as

$$c \stackrel{r}{\Rightarrow} c' \;\text{ iff }\; c \Rightarrow \langle g, u \rangle \stackrel{+}{\Rightarrow} c' \text{ for some configuration } \langle g, u \rangle \text{ with } g \in G.$$

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Definition (head, repeating head)

The head of a rule $\langle p, \gamma \rangle \hookrightarrow \langle p', w \rangle$ is the configuration $\langle p, \gamma \rangle$.

A head $\langle p, \gamma \rangle$ is repeating if $\langle p, \gamma \rangle \stackrel{r}{\Rightarrow} \langle p, \gamma v \rangle$ for some $v \in \Gamma^*$.
The set of repeating heads of $\mathcal{BP}$ is denoted by R.

Characterization of configurations with accepting runs

Lemma

Let c be a configuration of a Büchi pushdown system $\mathcal{BP}$.
$\mathcal{BP}$ has an accepting run starting from c $\iff$ there exists a repeating head $\langle p, \gamma \rangle$ such that $c \Rightarrow \langle p, \gamma w \rangle$ for some $w \in \Gamma^*$.

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Proof.

The implication "$\Longleftarrow$" is obvious. We prove "$\Longrightarrow$".

- assume that $\mathcal{BP}$ has an accepting run $\langle p_0, w_0 \rangle, \langle p_1, w_1 \rangle, \langle p_2, w_2 \rangle, \ldots$ starting from c
- let $i_0, i_1, \ldots$ be an increasing sequence of indices such that
  - $|w_{i_0}| = \min\{|w_j| \mid j \geq 0\}$
  - $|w_{i_k}| = \min\{|w_j| \mid j > i_{k-1}\}$ for $k > 0$
- once a configuration $\langle p_{i_k}, w_{i_k} \rangle$ is reached, the rest of the run never looks at or changes the bottom $|w_{i_k}| - 1$ stack symbols

Proof

Proof (Cont.)

- let γ_{i_k} be the topmost symbol of w_{i_k} for each $k \geq 0$
- as the number of pairs (p_{i_k}, γ_{i_k}) is bounded by $|P \times \Gamma|$, there has to be a pair (p, γ) repeated infinitely many times
- moreover, since some $g \in G$ becomes a control location infinitely often, we can select two indeces $j_1 < j_2$ out of $i_0, i_1, \ldots$ such that

$$\langle p_{j_1}, w_{j_1} \rangle = \langle p, \gamma w \rangle \ \stackrel{r}{\Rightarrow}\ \langle p_{j_2}, w_{j_2} \rangle = \langle p, \gamma v w \rangle$$

  for some $w, v \in \Gamma^*$
- as w is never looked at or changed in the rest of the run, we have that $\langle p, \gamma \rangle \stackrel{r}{\Rightarrow} \langle p, \gamma v \rangle$
- this proves "$\Longrightarrow$"

□

Consequences

Lemma

Let c be a configuration of a Büchi pushdown system $\mathcal{BP}$.
$\mathcal{BP}$ has an accepting run starting from $c \iff$ there exists a repeating head $\langle p, \gamma \rangle$ such that $c \Rightarrow \langle p, \gamma w \rangle$ for some $w \in \Gamma^*$.

- the set of all configurations violating the considered formula φ can be computed as $pre^*(R\Gamma^*)$, where $R\Gamma^* = \{\langle p, \gamma w \rangle \mid \langle p, \gamma \rangle \in R, w \in \Gamma^*\}$
- as R is finite, $R\Gamma^*$ is clearly regular
- $pre^*(C)$ can be easily computed for regular sets C
- the only remaining step to solve the model checking problem is the algorithm computing R

Computing R

Computing R is reduced to a graph-theoretic problem.

Given a $\mathcal{BP} = (P, \Gamma, \Delta, G)$, we construct a graph $\mathcal{G} = (P \times \Gamma, E)$ representing the reachability relation between heads, i.e.

- nodes are the heads of $\mathcal{BP}$,
- $E \subseteq (P \times \Gamma) \times \{0, 1\} \times (P \times \Gamma)$ is the smallest relation satisfying the following rule:

Rule

If $\langle p, \gamma \rangle \hookrightarrow \langle p'', v_1 \gamma' v_2 \rangle$ and $\langle p'', v_1 \rangle \Rightarrow \langle p', \varepsilon \rangle$ then

1. $((p, \gamma), 1, (p', \gamma')) \in E$ if $\langle p'', v_1 \rangle \stackrel{r}{\Rightarrow} \langle p', \varepsilon \rangle$ or $p \in G$
2. $((p, \gamma), 0, (p', \gamma')) \in E$ otherwise.

Computing R

Rule

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2. $((p, \gamma), 0, (p', \gamma')) \in E$ otherwise.

Edges are labelled with 1 if an accepting control state is passed between the heads, by 0 otherwise.

Conditions $\langle p'', v_1 \rangle \Rightarrow \langle p', \varepsilon \rangle$ or $\langle p'', v_1 \rangle \overset{r}{\Rightarrow} \langle p', \varepsilon \rangle$ can be checked by the algorithm for $pre^*(\{\langle p', \varepsilon \rangle\})$ or its small modification, respectively.

Computing R

Once $\mathcal{G}$ is constructed, R can be computed using the fact that:

a head $\langle p, \gamma \rangle$ is repeating $\iff$ (p, γ) is in a strongly connected component of $\mathcal{G}$ which has an internal edge labelled with 1

Example

Construct the graph $\mathcal{G}$ for $\mathcal{BP} = (\{p_0, p_1, p_2\}, \{\gamma_0, \gamma_1, \gamma_2\}, \Delta, \{p_2\})$, where

$$\Delta = \{ \quad \langle p_0, \gamma_0 \rangle \hookrightarrow \langle p_1, \gamma_1\gamma_0 \rangle, \quad \langle p_2, \gamma_2 \rangle \hookrightarrow \langle p_0, \gamma_1 \rangle, \\ \langle p_1, \gamma_1 \rangle \hookrightarrow \langle p_2, \gamma_2\gamma_0 \rangle, \quad \langle p_0, \gamma_1 \rangle \hookrightarrow \langle p_0, \varepsilon \rangle \quad \}.$$

Rule

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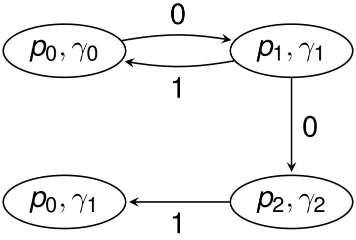

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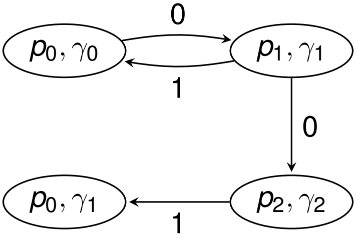
$$\Delta = \{ \quad \begin{array}{ll} \langle p_0, \gamma_0 \rangle \hookrightarrow \langle p_1, \gamma_1\gamma_0 \rangle, & \langle p_2, \gamma_2 \rangle \hookrightarrow \langle p_0, \gamma_1 \rangle, \\ \langle p_1, \gamma_1 \rangle \hookrightarrow \langle p_2, \gamma_2\gamma_0 \rangle, & \langle p_0, \gamma_1 \rangle \hookrightarrow \langle p_0, \varepsilon \rangle \end{array} \quad \}.$$



repeating heads: $\langle p_0, \gamma_0 \rangle$, $\langle p_1, \gamma_1 \rangle$

Algorithm: notes

We give an algorithm computing R for a given $\mathcal{BP}$ in normal form.

The algorithm runs in two phases.

1. It computes $\mathcal{A}_{pre^*}$ recognizing $pre^*(\{\langle p, \varepsilon\rangle \mid p \in P\})$. Every transition (p, γ, p') of $\mathcal{A}_{pre^*}$ signifies that $\langle p, \gamma\rangle \Rightarrow \langle p', \varepsilon\rangle$.

   We enrich the transitions of $\mathcal{A}_{pre^*}$: transitions (p, γ, p') are replaced by $(p, [\gamma, b], p')$ where b is a Boolean. The meaning of $(p, [\gamma, 1], p')$ should be that $\langle p, \gamma\rangle \stackrel{r}{\Rightarrow} \langle p', \varepsilon\rangle$.
2. It constructs the graph $\mathcal{G}$, identifies its strongly connected components (e.g. using Tarjan's algorithm), and determines the set of repeating heads.

We define $G(p) = 1$ if $p \in G$ and $G(p) = 0$ otherwise.

Algorithm

input : $\mathcal{BP} = (P, \Gamma, \Delta, G)$ in normal form
output: the set of repeating heads in $\mathcal{BP}$

rel $\leftarrow \emptyset$; trans $\leftarrow \emptyset$; $\Delta' \leftarrow \emptyset$
forall $\langle p, \gamma \rangle \hookrightarrow \langle p', \varepsilon \rangle \in \Delta$ **do** trans $\leftarrow$ trans $\cup \{(p, [\gamma, G(p)], p')\}$
while trans $\neq \emptyset$ **do**
  pop $t = (p, [\gamma, b], p')$ from trans
  if $t \notin$ rel **then**
    rel $\leftarrow$ rel $\cup \{t\}$
    forall $\langle p_1, \gamma_1 \rangle \hookrightarrow \langle p, \gamma \rangle \in \Delta$ **do** trans $\leftarrow$ trans $\cup \{(p_1, [\gamma_1, b \vee G(p_1)], p')\}$
    forall $\langle p_1, \gamma_1 \rangle \xhookrightarrow{b'} \langle p, \gamma \rangle \in \Delta'$ **do** trans $\leftarrow$ trans $\cup \{(p_1, [\gamma_1, b \vee b'], p')\}$
    forall $\langle p_1, \gamma_1 \rangle \hookrightarrow \langle p, \gamma\gamma_2 \rangle \in \Delta$ **do**
      $\Delta' \leftarrow \Delta' \cup \{\langle p_1, \gamma_1 \rangle \xhookrightarrow{b \vee G(p_1)} \langle p', \gamma_2 \rangle\}$
      forall $(p', [\gamma_2, b'], p'') \in$ rel **do**
        trans $\leftarrow$ trans $\cup \{(p_1, [\gamma_1, b \vee b' \vee G(p_1)], p'')\}$ `// end of part 1`
$R \leftarrow \emptyset$; $E \leftarrow \emptyset$ `// beginning of part 2`
forall $\langle p, \gamma \rangle \hookrightarrow \langle p', \gamma' \rangle \in \Delta$ **do** $E \leftarrow E \cup \{((p, \gamma), G(p), (p', \gamma'))\}$
forall $\langle p, \gamma \rangle \xhookrightarrow{b} \langle p', \gamma' \rangle \in \Delta'$ **do** $E \leftarrow E \cup \{((p, \gamma), b, (p', \gamma'))\}$
forall $\langle p, \gamma \rangle \hookrightarrow \langle p', \gamma'\gamma'' \rangle \in \Delta$ **do** $E \leftarrow E \cup \{((p, \gamma), G(p), (p', \gamma'))\}$
find all strongly connected components in $\mathcal{G} = ((P \times \Gamma), E)$
forall components C **do**
  if C has a 1-edge **then** $R \leftarrow R \cup C$
return R

Theorem

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Let $\mathcal{BP} = (P, \Gamma, \Delta, G)$ *be a Büchi pushdown system. The set of repeating heads* R *can be computed in* $\mathcal{O}(|P|^2 \cdot |\Delta|)$ *time and* $\mathcal{O}(|P| \cdot |\Delta|)$ *space.*

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Proof.

The first part is similar to the algorithm computing $\mathcal{A}_{pre^*}$.
The size of $\mathcal{G}$ is in $\mathcal{O}(|P| \cdot |\Delta|)$. Determining the strongly connected components takes linear time in the size of the graph [Tarjan1972]. The same holds for searching each component for an internal 1-edge. □

Theorem

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Let $\mathcal{P}$ be a pushdown system and φ be an LTL formula. The global model checking problem can be solved in $\mathcal{O}(|\mathcal{P}|^3 \cdot |\mathcal{A}|^3)$ time and $\mathcal{O}(|\mathcal{P}|^2 \cdot |\mathcal{A}|^2)$ space, where $\mathcal{A}$ is a Büchi automaton corresponding to $\neg\varphi$.