IA169 Model Checking

Abstraction and CEGAR

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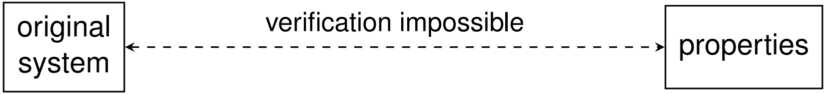
Motivation

Abstraction is one of the most important techniques for reducing the state explosion problem.

[CGKPV18]



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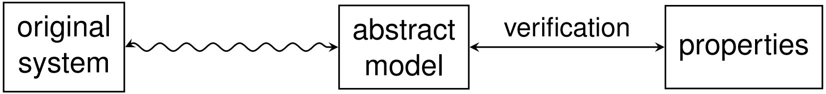

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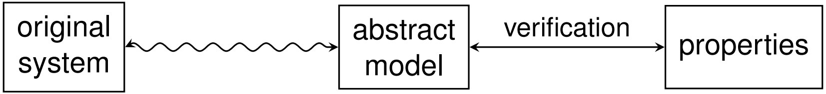
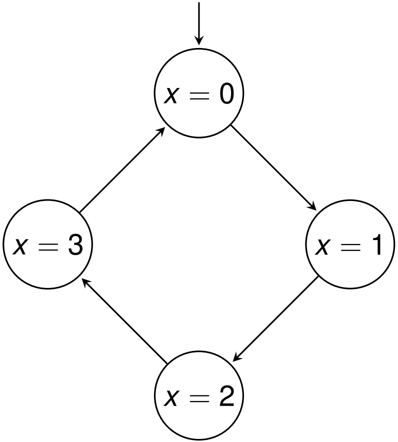


- large finite systems $\longrightarrow$ smaller finite systems
- infinite-state systems $\longrightarrow$ finite systems

Intuition



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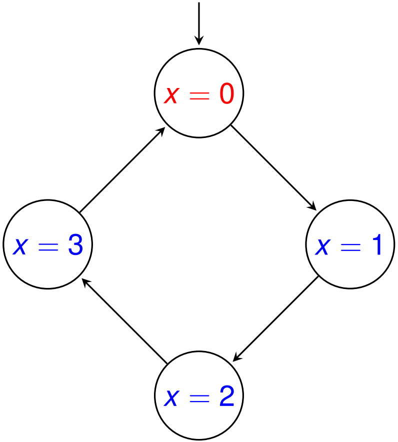

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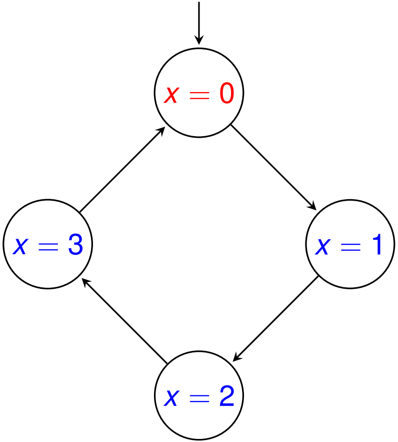

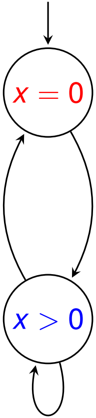

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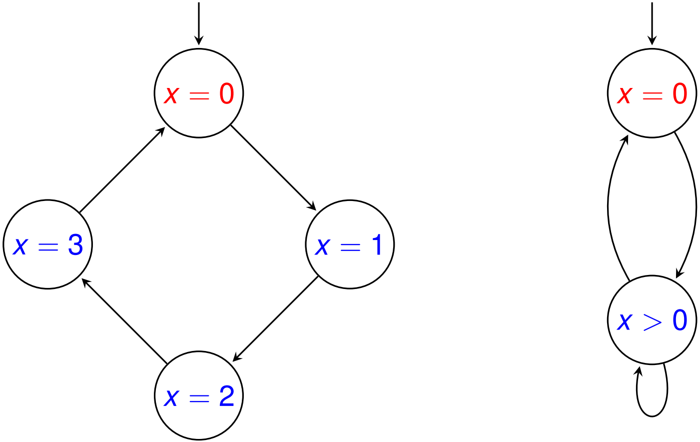

- equivalent with respect to $\mathsf{F}(x > 0)$
- nonequivalent with respect to $\mathsf{GF}(x = 0)$

Agenda and sources

agenda

- simulation
- exact abstractions
- non-exact abstractions, in particular predicate abstraction
- abstraction in practice
- CEGAR: counterexample-guided abstraction refinement

sources

- Chapter 13 of E. M. Clarke, O. Grumberg, D. Kroening, D. Peled, and H. Veith: *Model Checking*, Second Edition, MIT, 2018.
- R. Pelánek: *Reduction and Abstraction Techniques for Model Checking*, PhD thesis, FI MU, 2006.
- E. M. Clarke, O. Grumberg, S. Jha, Y. Lu, H. Veith: *Counterexample-guided Abstraction Refinement for Symbolic Model Checking*, J. ACM 50(5), 2003.

Simulation

Simulation

Definition (simulation)

Given two Kripke structures $M = (S, \to, S_0, L)$ and $M' = (S', \to', S'_0, L')$, we say that M' simulates M, written $M \leq M'$, if there exists a relation $R \subseteq S \times S'$ such that:

- $\forall s_0 \in S_0 \,.\, \exists s'_0 \in S'_0 \,.\, (s_0, s'_0) \in R$
- $(s, s') \in R \implies L(s) = L'(s')$
- $(s, s') \in R \wedge s \to p \implies \exists p' \in S' \,.\, s' \to' p' \wedge (p, p') \in R$

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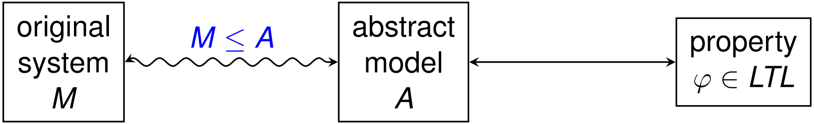
Lemma

If $M \leq M'$, then for every path $\sigma = s_1 s_2 \ldots$ of M starting in an initial state there is a run $\sigma' = s'_1 s'_2 \ldots$ of M' starting in an initial state and satisfying

$$L(s_1)L(s_2)\ldots = L'(s'_1)L'(s'_2)\ldots.$$

Relations between original and abstract systems



$M \leq A \implies$ all behaviours of M are also in A
(but not vice versa)

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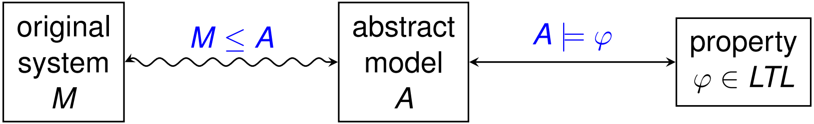

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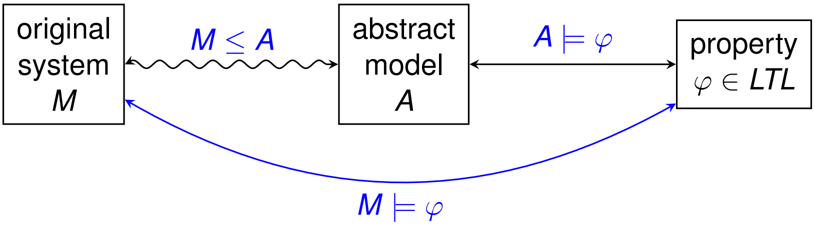

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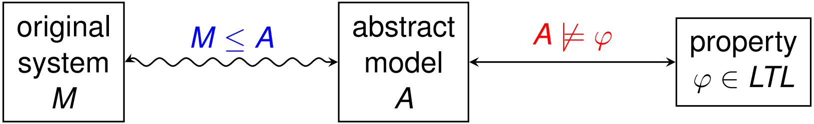

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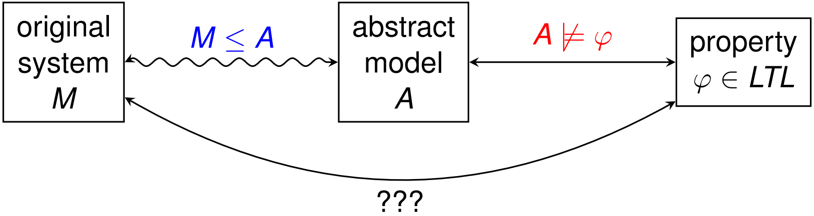

If A has a behaviour violating φ (i.e. $A \not\models \varphi$), then either

1. M has this behaviour as well (i.e. $M \not\models \varphi$), or
2. M does not have this behaviour, which is then called false positive or spurious counterexample ($M \models \varphi$ or $M \not\models \varphi$ due to another behaviour violating φ).

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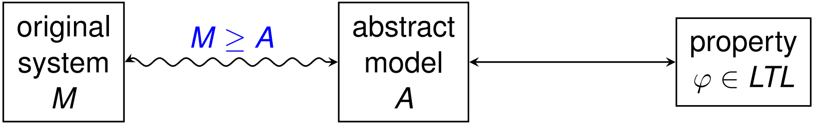

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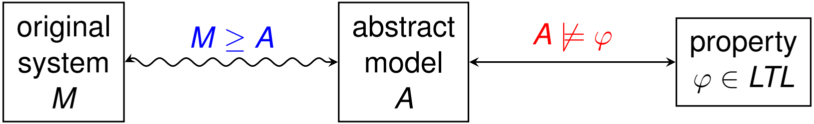

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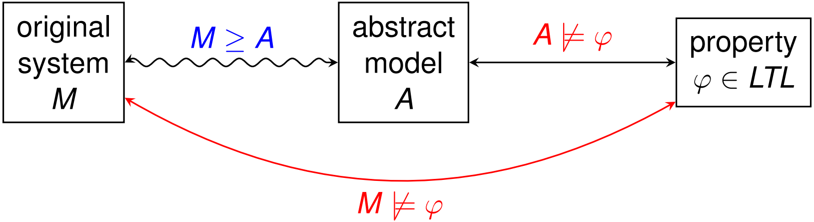

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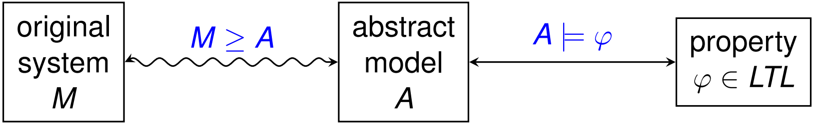

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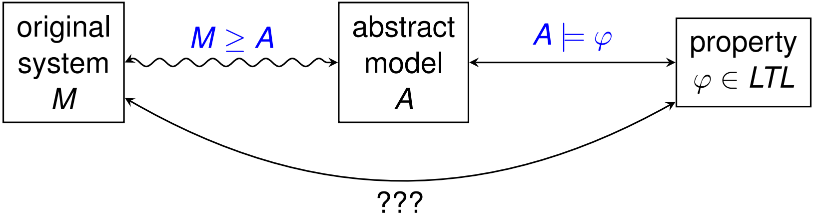

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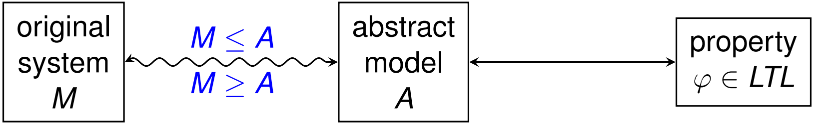

$M \leq A \leq M \implies$ A and M have tha same behaviours
A is an exact abstraction of M

note: A and M are bisimilar $\implies$ $M \leq A \leq M$
$\not\Longleftarrow$

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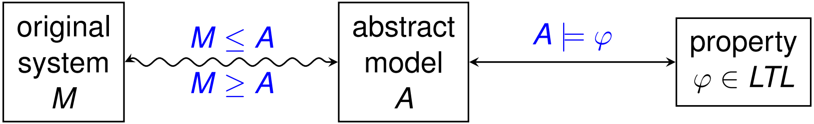

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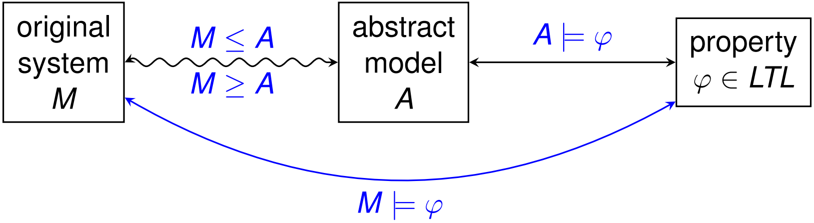

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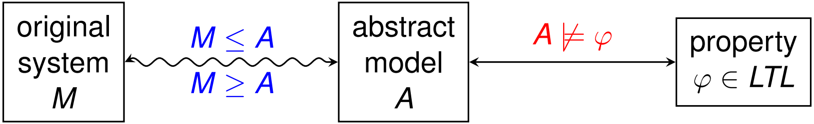

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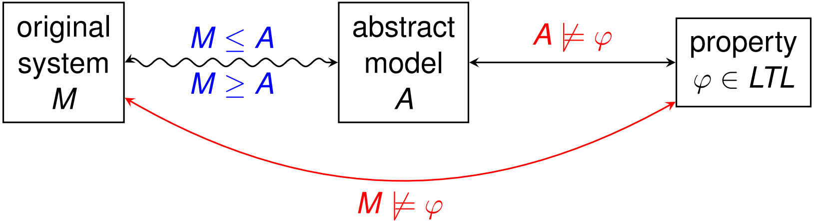

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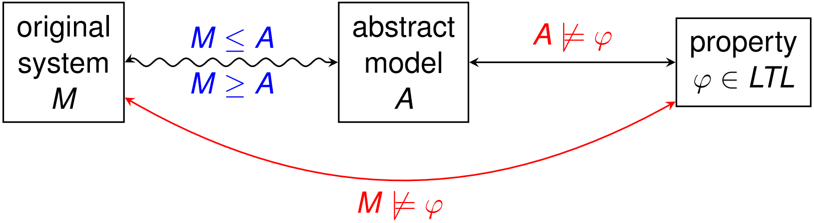

All these relations hold even for $\varphi \in$ ACTL*, where ACTL* is a fragment of CTL* without any existential quantifier (and negation above quantifiers).

Exact abstractions

Cone of influence (aka dead variables)

Idea

We eliminate the state variables that do not influence the variables in the specification.

Cone of influence (aka dead variables)

- assume that our system is a program
- let V be the set of variables appearing in specification
- cone of influence C of V is the minimal set of variables such that
  - $V \subseteq C$
  - if v occurs in a test affecting the control flow, then $v \in C$
  - if there is an assignment $v := e$ for some $v \in C$, then all variables occurring in the expression e are also in C
- C can be computed by the source code analysis
- variables that are not in C can be eliminated from the code together with all commands they participate in

Cone of influence: example

```
S: v = getinput();
   x = getinput();
   y = 1;
   z = 1;
   while (v > 0) {
     z = z * x;
     x = x - 1;
     y = y * v;
     v = v - 1;
   }
   z = z * y;
E:
```

specification: $\mathsf{F}(pc = \mathtt{E})$

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   }
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E:
```

```
S: v = getinput();
   skip;
   skip;
   skip;
   while (v > 0) {
     skip;
     skip;
     skip;
     v = v - 1;
   }
   skip;
E:
```

specification: $\mathsf{F}(pc = \mathtt{E})$

$V = \emptyset$, $C = \{\mathtt{v}\}$

Other exact abstractions

symmetry reduction

- in systems with more identical parallel components, their order is not important

equivalent values

- if the set of behaviours starting in a state s is the same for values a, b of a variable v, then the two values can be replaced by one
- applicable to larger sets of values as well
- used in timed automata for timer values

Non-exact abstractions, in particular predicate abstraction

Concept

we face two problems

1. to find a suitable set of abstract states (called abstract domain) and a mapping between the original states and the abstract ones
2. to compute a transition relation on abstract states

Finding abstract states

abstract states are usually defined in one of the following ways

1. for each variable x, we replace the original variable domain D_x by an abstract variable domain A_x and we define a total function $h_x : D_x \to A_x$

   a state $s = (v_1, \ldots, v_m) \in D_{x_1} \times \ldots \times D_{x_m}$ given by values of all variables corresponds to an abstract state

   $$h(s) = (h_{x_1}(v_1), \ldots, h_{x_m}(v_m)) \in A_{x_1} \times \ldots \times A_{x_m}$$

2. predicate abstraction - we choose a finite set $\Phi = \{\phi_1, \ldots, \phi_n\}$ of predicates over the set of variables;
   we have several choices of an abstract domain

The first approach can be seen as a special case the latter one.

Popular abstract domains for integers

sign abstraction

- $A_x = \{a_+, a_-, a_0\}$
- $h_x(v) = \begin{cases} a_- & \text{if } v < 0 \\ a_0 & \text{if } v = 0 \\ a_+ & \text{if } v > 0 \end{cases}$

parity abstraction

- $A_x = \{a_e, a_o\}$
- $h_x(v) = \begin{cases} a_e & \text{if } v \text{ is even} \\ a_o & \text{if } v \text{ is odd} \end{cases}$
- good for verification of properties related to the last bit of binary representation

Popular abstract domains for integers

congruence modulo an integer

- $A_x = \{0, 1, \ldots, m-1\}$ for some $m > 1$
- $h_x(v) = v \bmod m$
- nice properties

$$\begin{aligned}
((x \bmod m) + (y \bmod m)) \bmod m &= (x + y) \bmod m \\
((x \bmod m) - (y \bmod m)) \bmod m &= (x - y) \bmod m \\
((x \bmod m) \cdot (y \bmod m)) \bmod m &= (x \cdot y) \bmod m
\end{aligned}$$

representation by logarithm

- $h_x(v) = \lceil \log_2(v + 1) \rceil$
- the number of bits needed to represent v
- good for verification of properties related to overflow problems

Popular abstract domains for integers

single bit abstraction

- $A_x = \{0, 1\}$
- $h_x(v) =$the i-th bit of v for a fixed i

single value abstraction

- $A_x = \{0, 1\}$
- $h_x(v) = \begin{cases} 1 & \text{if } v = c \\ 0 & \text{otherwise} \end{cases}$

...and others

Predicate abstraction

Let $\Phi = \{\phi_1, \dots, \phi_n\}$ be a set of predicates over the set of variables.

abstract domain $\{0, 1\}^n$

- a state $s = (v_1, \dots, v_m)$ corresponds to an abstract state given by a vector of truth values of $\{\phi_1, \dots, \phi_n\}$, i.e.,

$$h(s) = (\phi_1(v_1, \dots, v_m), \dots, \phi_n(v_1, \dots, v_m)) \in \{0, 1\}^n$$

- example: $\phi_1 = (x_1 > 3) \quad \phi_2 = (x_1 < x_2) \quad \phi_3 = (x_2 > 10)$
  $s = (5, 7)$
  $h(s) = (1, 1, 0)$

Abstract structures

assume that

- we have an original Kripke structure $M = (S, \to, S_0, L)$
- we have an abstract domain A and a mapping $h : S \to A$

we define an abstract model as a Kripke structure $(A, \to', A_0, L_A)$, where

- $A_0 = \{h(s_0) \mid s_0 \in S_0\}$
- $L_A : A \to 2^{AP}$ has to be correctly defined, i.e.,
  - for abstraction based on variable domains, validity of atomic propositions is determined by abstract states in $A_{x_1} \times \ldots \times A_{x_m}$
  - for predicate abstraction, validity of atomic propositions is determined by abstraction predicates $\{\phi_1, \ldots, \phi_n\}$ (AP is typically a subset of it)

  and L_A has to agree with L, i.e., $L(s) = L_A(h(s))$
- $\to'$ is defined in one of the following ways

May abstraction

may abstraction produces $M_{may} = (A, \rightarrow_{may}, A_0, L_A)$

- $a_1 \rightarrow_{may} a_2$ iff there exist $s_1, s_2 \in S$ such that $h(s_1) = a_1$, $h(s_2) = a_2$, $s_1 \rightarrow s_2$

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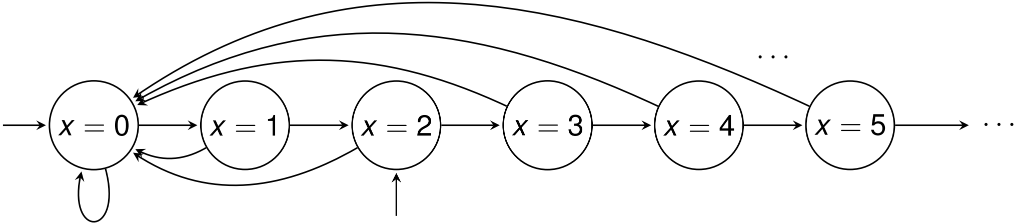
- example: construct M_{may} for the following system using predicate abstraction with predicates $\phi_1 = (x > 0)$ and $\phi_2 = (x > 2)$ and abstract domain $\{0, 1\}^2$



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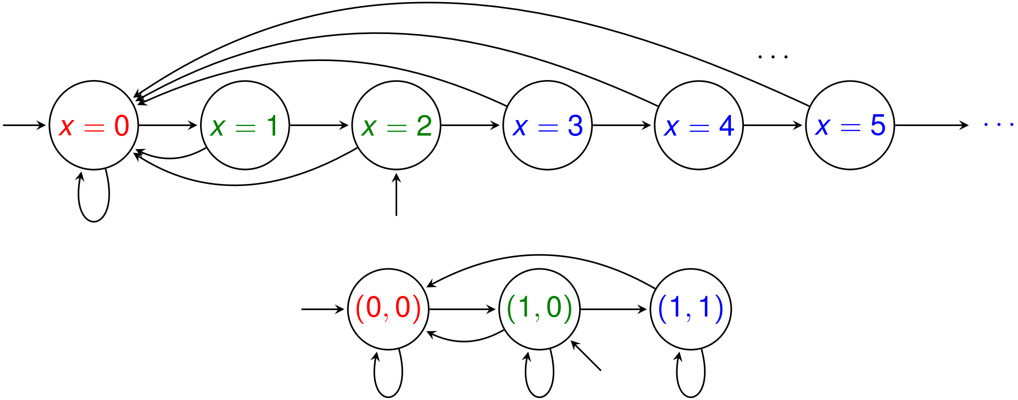

Must abstraction

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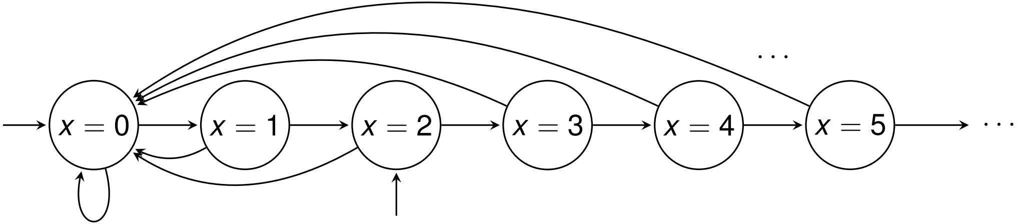

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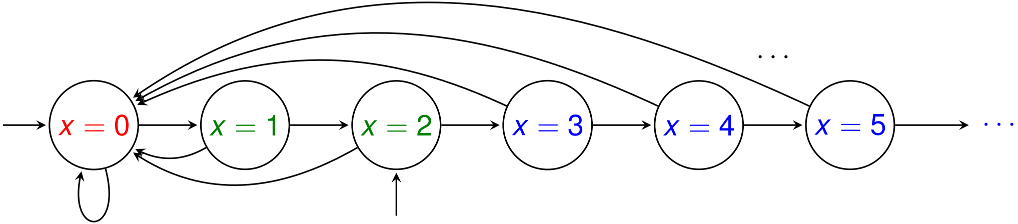

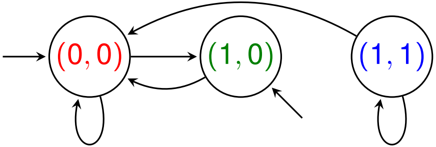

Relations between M, M_{must}, and M_{may}

Lemma

For every Kripke structure M, abstract domain A with a mapping function h it holds

$$M_{must} \;\leq\; M \;\leq\; M_{may}.$$

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For every Kripke structure M, abstract domain A with a mapping function h it holds

$$M_{must} \;\leq\; M \;\leq\; M_{may}.$$

- computing M_{must} or M_{may} requires constructing M first (recall that M can be very large or even infinite)
- we rather compute an under-approximation M'_{must} of M_{must} or an over-approximation M'_{may} of M_{may} directly from the implicit representation of M
- it holds that $M'_{must} \leq M_{must} \leq M \leq M_{may} \leq M'_{may}$

Abstraction in practice

Predicate abstraction: abstracting sets of states

Abstract domain $\{0,1\}^n$ can lead to too many transitions $\implies$ it is sometimes better to assign a single abstract state to a set of original states.

abstract domain $2^{\{0,1\}^n}$

- let $\vec{b} = \langle b_1, \ldots, b_n \rangle$ be a vector of $b_i \in \{0,1\}$
- we set $[\vec{b}, \Phi] = b_1 \cdot \phi_1 \wedge \ldots \wedge b_n \cdot \phi_n$, where $0 \cdot \phi_i = \neg\phi_i$ and $1 \cdot \phi_i = \phi_i$
- let X denote the set of original states
- $h(X) = \{\vec{b} \in \{0,1\}^n \mid \exists s \in X : s \models [\vec{b}, \Phi]\}$
- example: $\phi_1 = (x_1 > 3) \quad \phi_2 = (x_1 < x_2) \quad \phi_3 = (x_2 > 10)$
  $X = \{(5,7), (4,5), (2,9)\}$
  $h(X) = \{(1,1,0), (0,1,0)\}$
- nice theoretical properties
- not used in practice (this abstract domain grows too fast)

Predicate abstraction: abstracting sets of states

abstract domain $\{0, 1, *\}^n$ (predicate-cartesian abstraction)

- let $\vec{b} = \langle b_1, \ldots, b_n \rangle$ be a vector of $b_i \in \{0, 1, *\}$
- we set $[\vec{b}, \Phi] = b_1 \cdot \phi_1 \wedge \ldots \wedge b_n \cdot \phi_n$, where $0 \cdot \phi_i = \neg\phi_i$, $1 \cdot \phi_i = \phi_i$, $* \cdot \phi_i = \top$
- $h(X) = \min\{\vec{b} \in \{0, 1, *\}^n \mid \forall s \in X : s \models [\vec{b}, \Phi]\}$, where $\min$ means "the most specific"
- example: $\phi_1 = (x_1 > 3) \quad \phi_2 = (x_1 < x_2) \quad \phi_3 = (x_2 > 10)$
  $X = \{(5, 7), (4, 5), (2, 9)\}$
  $h(X) = (*, 1, 0)$
- this one is sometimes used in practice

Guarded command language

syntax

- let V be a finite set of integer variables
- Act is a set of action names
- model is a pair $M = (V, E)$, where $E = \{t_1, \ldots, t_m\}$ is a finite set of transitions of the form $t_i = (a_i, g_i, u_i)$, where
  - $a_i \in Act$
  - g_i is a first-order formula called guard and built with V, integers, standard binary operations $(+, -, \cdot, \ldots)$ and relations $(=, <, >, \ldots)$
  - u_i is a finite sequence of assignments $x := e$, where $x \in V$ and e is an expression built with V, integers, and standard binary operations $(+, -, \cdot, \ldots)$

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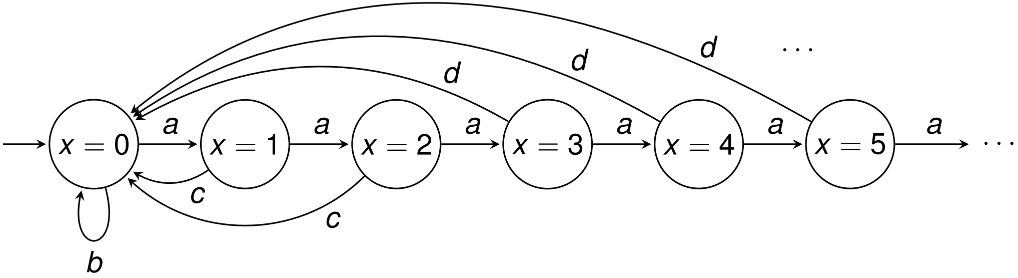
semantics

- M defines a labelled transition system where
  - states are valuations of variables $S = 2^{V \to \mathbb{Z}}$
  - initial state is the zero valuation $s_0(v) = 0$ for all $v \in V$
  - $s \stackrel{a_i}{\to} s'$ whenever $s \models g_i$ and s' arises from s by applying the assignments in u_i
- M can also describe a Kripke structure if we add a labelling function

Example



implicit description in guarded command language by model (V, E), where

$$
\begin{aligned}
V &= \{x\} \\
E &= \{(a,\ \top,\quad x := x + 1), \\
&\quad (b,\ \neg(x > 0),\quad x := 0), \\
&\quad (c,\ (x > 0) \wedge (x \leq 2),\quad x := 0), \\
&\quad (d,\ (x > 2),\quad x := 0)\}
\end{aligned}
$$

Abstraction in practice

- we use predicate abstraction with domain $\{0, 1, *\}^n$
- given a formula φ with free variables $\vec{x}$ from V, we set

$$pre(a_i, \varphi) = (g_i \implies \varphi[\vec{x}/u_i(\vec{x})])$$

  where $\varphi[\vec{x}/u_i(\vec{x})]$ denotes the formula φ with each free variable x replaced by $u_i(x)$, which is the expression representing the value of x after the assignments in u_i
- intuitively, $pre(a_i, \varphi)$ transforms the condition φ to the situation before taking the transition (a_i, g_i, u_i)
- we use a sound (potentially not complete) decision procedure *is_valid*, i.e.,

$$is\_valid(\varphi) = \top \implies \varphi \text{ is a tautology}$$

Abstraction in practice

for every abstract state $\vec{b} \in \{0, 1, *\}^n$ and for every transition $t_i = (a_i, g_i, u_i)$, we compute an over-approximation of a *may*-successor of $\vec{b}$ under t_i as

- if $\mathit{is\_valid}([\vec{b}, \Phi] \implies \neg g_i)$ then there is no successor
- otherwise, the successor $\vec{b'}$ is given by

$$b'_j = \begin{cases} 1 & \text{if } \mathit{is\_valid}([\vec{b}, \Phi] \implies \mathit{pre}(a_i, \phi_j)) \\ 0 & \text{if } \mathit{is\_valid}([\vec{b}, \Phi] \implies \mathit{pre}(a_i, \neg\phi_j)) \\ * & \text{otherwise} \end{cases}$$

Abstraction in practice

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- example: consider the abstract state $\vec{b} = (1, 0)$ where $\phi_1 = (x > 0)$ and $\phi_2 = (x > 2)$ and compute the successor corresponding to $(a, \top, x := x + 1)$

$$(1, 0) \xrightarrow{a}_{may'} (\quad , \quad)$$

Abstraction in practice

for every abstract state $\vec{b} \in \{0, 1, *\}^n$ and for every transition $t_i = (a_i, g_i, u_i)$, we compute an over-approximation of a *may*-successor of $\vec{b}$ under t_i as

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  - $(x > 0) \wedge (x \leq 2) \implies (\top \implies (x + 1 > 0))$ is true

Abstraction in practice

for every abstract state $\vec{b} \in \{0,1,*\}^n$ and for every transition $t_i = (a_i, g_i, u_i)$, we compute an over-approximation of a *may*-successor of $\vec{b}$ under t_i as

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- example: consider the abstract state $\vec{b} = (1,0)$ where $\phi_1 = (x > 0)$ and $\phi_2 = (x > 2)$ and compute the successor corresponding to $(a, \top, x := x + 1)$

$$(1,0) \xrightarrow{a}_{may'} (1, *)$$

  - $(x > 0) \wedge (x \leq 2) \implies (\top \implies (x + 1 > 0))$ is true
  - $(x > 0) \wedge (x \leq 2) \implies (\top \implies (x + 1 > 2))$ is not true
  - $(x > 0) \wedge (x \leq 2) \implies (\top \implies (x + 1 \leq 2))$ is not true

Abstraction in practice

- for every transition, we compute successors of all abstract states
- based on the successors, we transform the original implicit representation of a system into a Boolean program
- it is very similar to a model in guarded command language, but instead of integers it uses only Boolean variables $\vec{b}$ representing the validity of abstraction predicates Φ
- Boolean program is an implicit representation of an over-approximation of M_{may}
- Boolean program can be used as an input for a suitable model checker (of finite-state systems)

Example

- consider the model (V, E), where

$$
\begin{aligned}
V &= \{x\} \\
E &= \{(a,\ \top, & x := x + 1), \\
&\quad (b,\ \neg(x > 0), & x := 0), \\
&\quad (c,\ (x > 0) \wedge (x \leq 2), & x := 0), \\
&\quad (d,\ (x > 2), & x := 0)\}
\end{aligned}
$$

- using the predicates $\phi_1 = (x > 0)$, $\phi_2 = (x > 2)$, we get the following Boolean program defining an over-approximation of M_{may}

$$
\begin{aligned}
V &= \{b_1, b_2\}, \text{ where } b_1, b_2 \text{ represents the validity of } \phi_1, \phi_2 \\
E &= \{(a,\ \top, & b_1 := \mathit{if}\ b_1\ \mathit{then}\ 1\ \mathit{else}\ *; \\
& & b_2 := \mathit{if}\ b_2\ \mathit{then}\ 1\ \mathit{else}\ \mathit{if}\ b_1\ \mathit{then}\ *\ \mathit{else}\ 0), \\
&\quad (b,\ \neg b_1, & b_1 := 0;\ b_2 := 0), \\
&\quad (c,\ b_1 \wedge \neg b_2, & b_1 := 0;\ b_2 := 0), \\
&\quad (d,\ b_2, & b_1 := 0;\ b_2 := 0)\}
\end{aligned}
$$

Example of a real NQC code and its absraction

```
task light_sensor_control() {
  int x = 0;
  while (true) {
    if (LIGHT > LIGHT_THRESHOLD) {
      PlaySound(SOUND_CLICK);
      Wait(30);
      x = x + 1;
    } else {
      if (x > 2) {
        PlaySound(SOUND_UP);
        ClearTimer(0);
        brick = LONG;
      } else if (x > 0) {
        PlaySound(SOUND_DOUBLE_BEEP);
        ClearTimer(0);
        brick = SHORT;
      }
      x = 0;
    }
  }
}
```

Example of a real NQC code and its absraction

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  while (true) {
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        brick = LONG;
      } else if (x > 0) {
        PlaySound(SOUND_DOUBLE_BEEP);
        ClearTimer(0);
        brick = SHORT;
      }
      x = 0;
    }
  }
}
```

```
task A_light_sensor_control() {
  bool b = false;
  while (true) {
    if (*) {


      b = b ? true : * ;
    } else {
      if (b) {


        brick = LONG;
      } else if (b ? true : *) {


        brick = SHORT;
      }
      b = false;
    }
  }
}
```

CEGAR: counterexample-guided abstraction refinement

Motivation

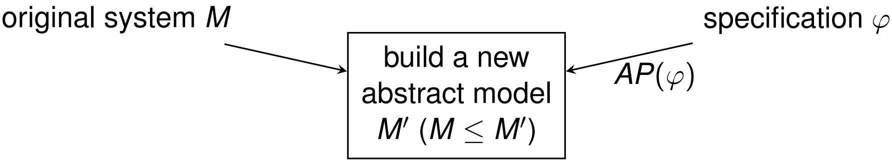
- it is hard to find a small and valuable abstraction
- abstraction predicates were originally provided by a user
- CEGAR tries to find a suitable abstraction automatically
- implemented in SLAM, BLAST, Static Driver Verifier (SDV), and many others
- incomplete method, but very successfull in practice

Principle of CEGAR

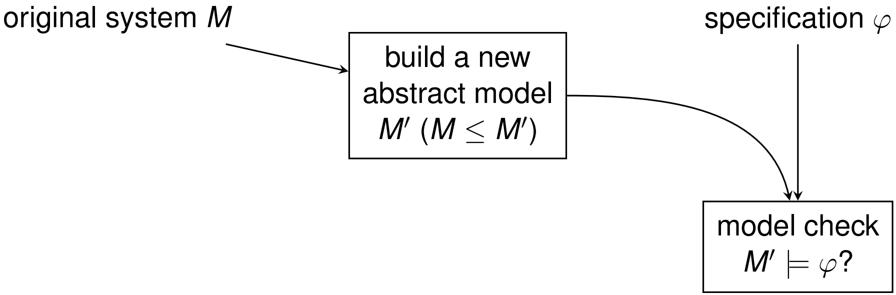
original system M

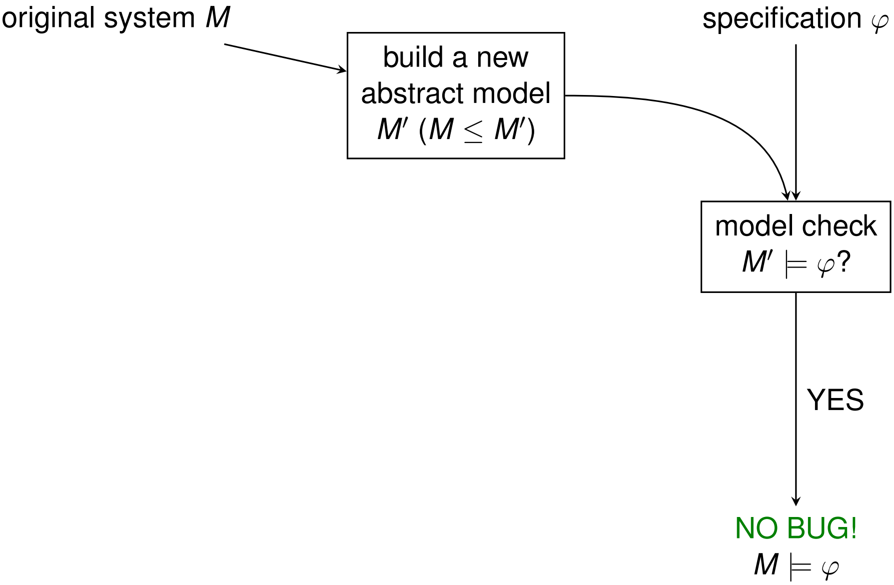
specification φ

Principle of CEGAR

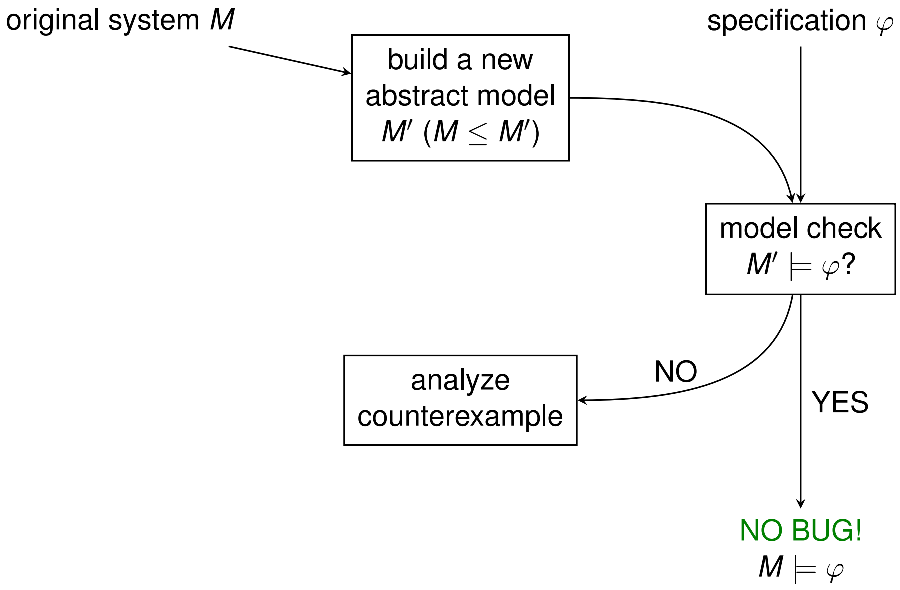


Principle of CEGAR




original system M
specification φ
build a new abstract model M' ($M \leq M'$)
model check $M' \models \varphi$?
YES
NO BUG!
$M \models \varphi$

Principle of CEGAR



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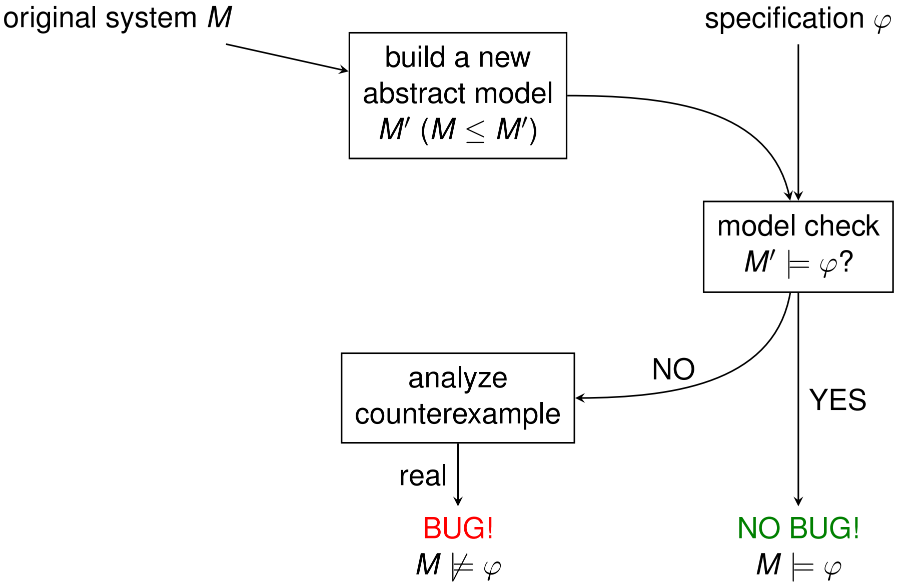

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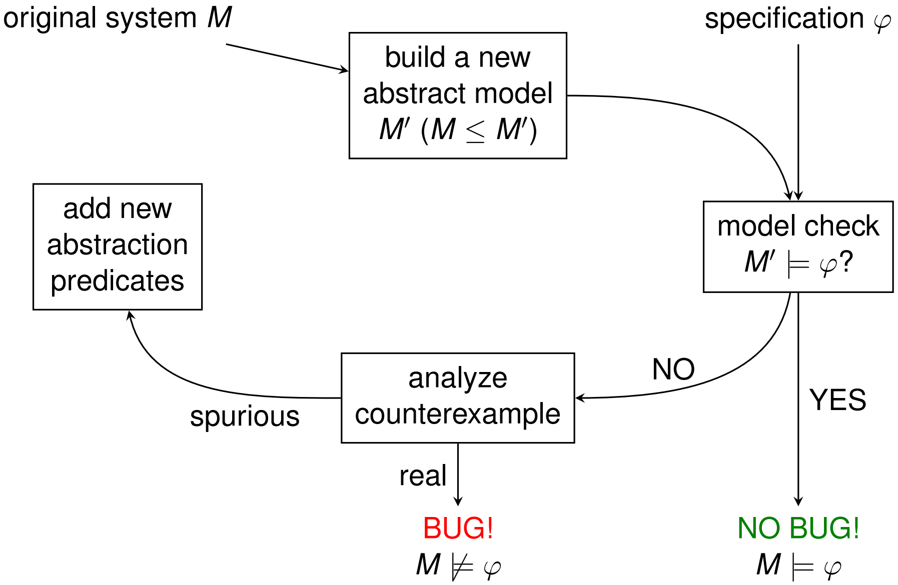

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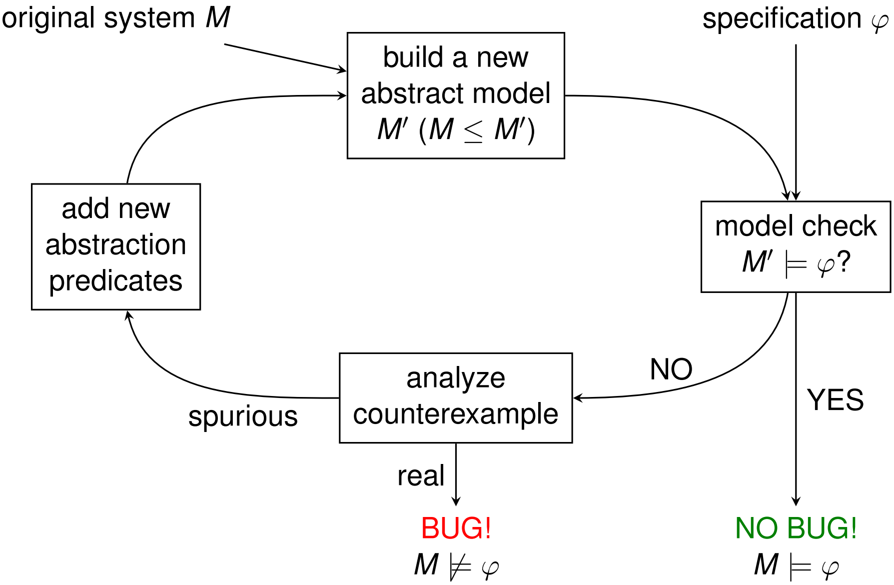
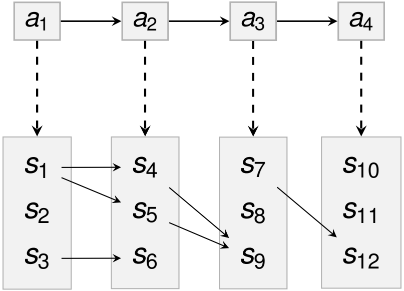


Notes

- added abstraction predicates ensure that the new abstract model M' does not have the behaviour corresponding to the spurious counterexample of the previous M'
- the analysis of an abstract counterexample and finding new abstract predicates are nontrivial tasks
- the method is sound but incomplete: the algorithm can run in the cycle forever or fail to find new abstraction predicates

Counterexample analysis

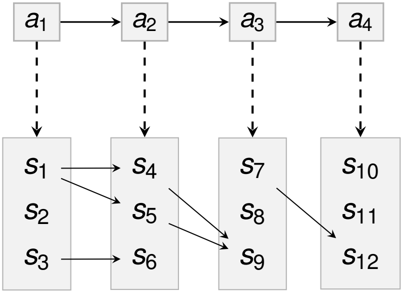
- an abstract path is a finite or infinite path in an abstract model
- an abstract path $a_1 a_2 \ldots$ is real if there exists a path $s_1 s_2 \ldots$ in the original system M of the same length such that s_1 is initial and $s_i \in h^{-1}(a_i)$ for all i
- an abstract path that is not real is called spurious



input : a nonempty abstract path $a_1 \dots a_n$,
an original system $M = (S, \to, S_0, L)$, an abstraction function h

output: "real" if the path is real; j, R' otherwise, where j is the length of the maximal real prefix of the path and R' is the set of the last states of the paths in M corresponding to the prefix

$R \leftarrow h^{-1}(a_1) \cap S_0$
if $R = \emptyset$ **then return** $0, \emptyset$ `// spurious`
$j \leftarrow 1$
while $R \neq \emptyset \wedge j < n$ **do**
  $j \leftarrow j + 1$
  $R' \leftarrow R$
  $R \leftarrow \{s \mid \exists s' \in R \,.\, s' \to s\} \cap h^{-1}(a_j)$
if $R \neq \emptyset$ **then return** real
return $j - 1, R'$ `// spurious`



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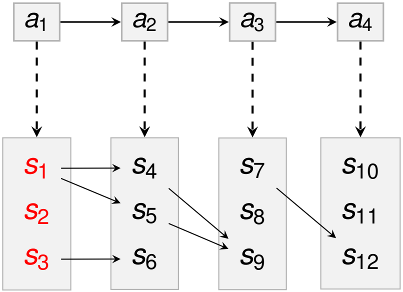

Analysis of finite counterexamples (e.g. for reachability)

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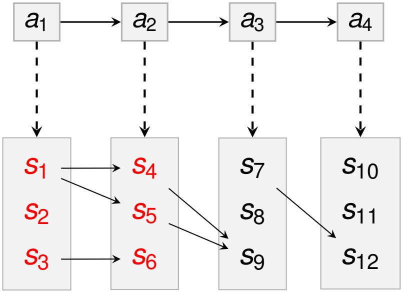

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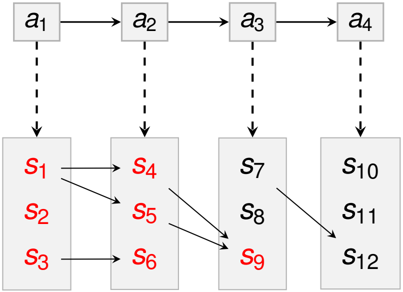
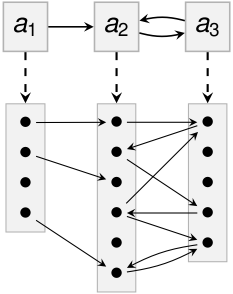


produced output: $3, \{s_9\}$

Analysis of lasso-shaped counterexamples



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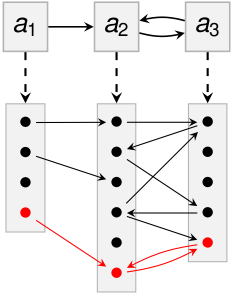

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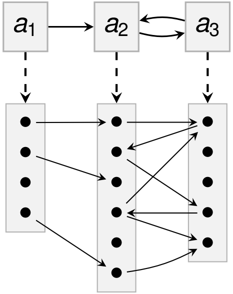

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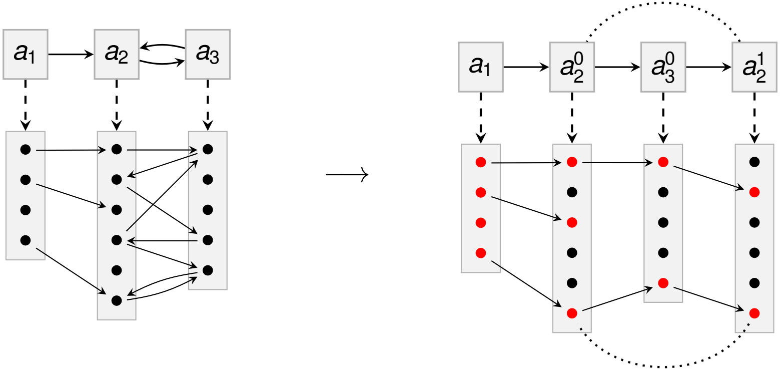
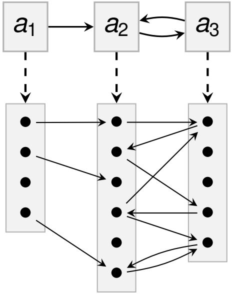
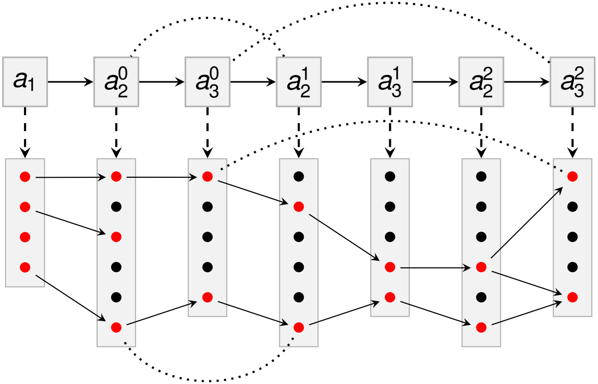


Analysis of lasso-shaped counterexamples



$\longrightarrow$



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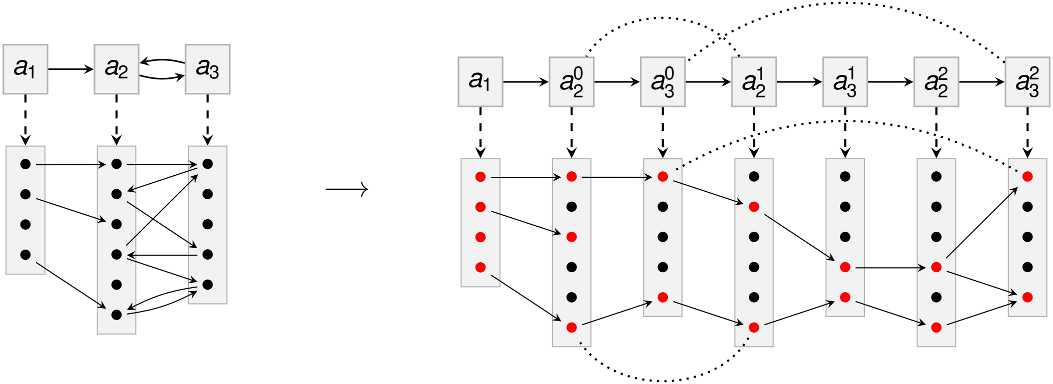

- an abstract loop may correspond to loops of different size and starting at different stages of the unwinding
- the unwinding eventually becomes periodic, the size of the period is the least common multiple of the size of individual loops

Analysis of lasso-shaped counterexamples

Analysis of a lasso-shaped counterexample can be reduced to analysis of a finite path counterexample.

Theorem

An abstract lasso-shaped path $a_1 \ldots a_i(a_{i+1} \ldots a_n)^\omega$ *is real iff the abstract path* $a_1 \ldots a_i(a_{i+1} \ldots a_n)^{m+1}$ *is real, where* $m = \mathrm{min}_{i+1 \leq j \leq n} |h^{-1}(a_j)|$.

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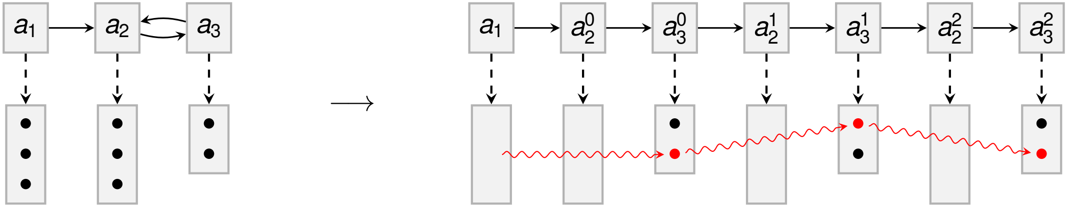

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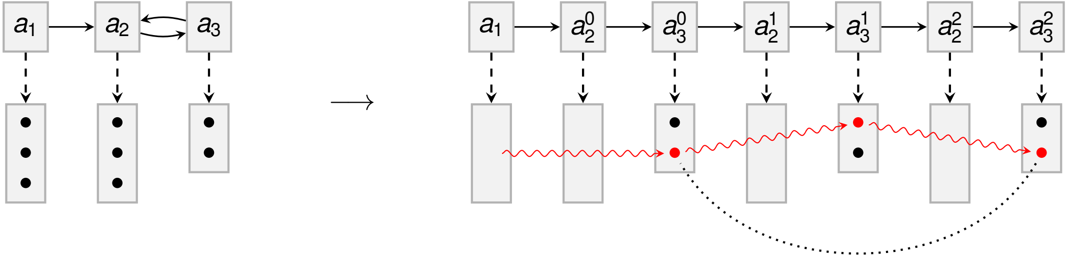
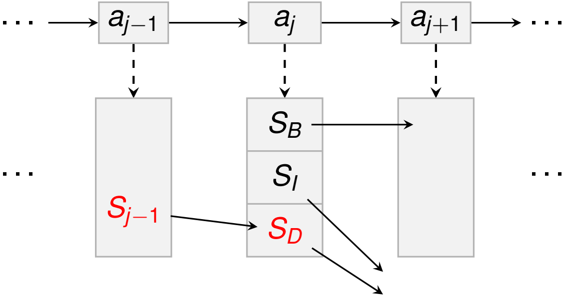


Abstraction refinement

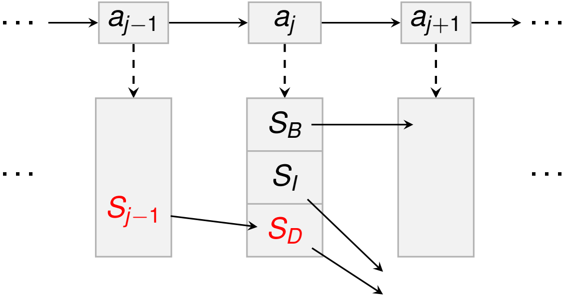


$$S_B = h^{-1}(a_j) \cap \{s \mid \exists s' \in h^{-1}(a_{j+1}) \,.\, s \to s'\}$$ bad states

$$S_I = h^{-1}(a_j) \setminus (S_B \cup S_D)$$ irrelevant states

S_D = the set produced by the counterexample analysis — dead-end states

Abstraction refinement



$$
\begin{aligned}
S_B &= h^{-1}(a_j) \cap \{s \mid \exists s' \in h^{-1}(a_{j+1}) \,.\, s \to s'\} && \text{bad states} \\
S_I &= h^{-1}(a_j) \setminus (S_B \cup S_D) && \text{irrelevant states} \\
S_D &= \text{the set produced by the counterexample analysis} && \text{dead-end states}
\end{aligned}
$$

- to eliminate the spurious counterexample, we need to refine the abstraction such that no abstract state contains states from both S_B and S_D
- typically, we add an abstraction predicate that is an interpolant of S_B and S_D

Abstraction refinement

Consider abstract state $(3 \leq x \leq 5) \land (7 \leq y \leq 9)$ and S_B, S_I, S_D:

	3	4	5
7	B	I	I
8	D	I	B
9	I	D	D

Abstraction refinement

Consider abstract state $(3 \leq x \leq 5) \wedge (7 \leq y \leq 9)$ and S_B, S_I, S_D:

	3	4	5
7	B	I	I
8	D	I	B
9	I	D	D

$\longrightarrow$

	3 4	5
7	B + I	I
8	D + I	B
9	D + I	D

or

	3	4 5
7, 9	B + I	D + I
8	D	B + I

?

- there could be more possible abstraction refinements
- we want the coarsest refinement (i.e., with the least number of abstract states)

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Theorem

The problem of finding the coarsest refinement is NP-hard.

- there are heuristics that select suitable refinements