IA008: Computational Logic

2. First-Order Logic

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Basic Concepts

First-Order Logic

Syntax

- variables $x, y, z, \ldots$
- terms x, $f(t_0, \ldots, t_n)$
- relations $R(t_0, \ldots, t_n)$ and equality $t_0 = t_1$
- operators $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
- quantifiers $\exists x \varphi$, $\forall x \varphi$

Semantics

$$\mathfrak{A} \models \varphi(\bar{a}) \qquad \mathfrak{A} = \langle A, R_0, R_1, \ldots, f_0, f_1, \ldots \rangle$$

Examples

$$\varphi := \forall x \exists y [f(y) = x],$$

$$\psi := \forall x \forall y \forall z [x \leq y \wedge y \leq z \rightarrow x \leq z].$$

Examples

Structures

- graphs $\mathfrak{G} = \langle V, E \rangle$

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  $P_a \subseteq W$ positions with letter a

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  $\leq \subseteq W \times W$ linear ordering

  $P_a \subseteq W$ positions with letter a

- transition systems $\mathfrak{S} = \langle S, (E_a)_a, (P_i)_i \rangle$

  $E_a \subseteq V \times V$ binary relation

  $P_i \subseteq V$ unary relation

Examples

Graphs $\mathfrak{G} = \langle V, E \rangle,\ E \subseteq V \times V$

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Examples

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- 'The graph has no isolated vertices.'

Examples

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$$\forall x \forall y [E(x, y) \to E(y, x)]$$

- 'The graph has no isolated vertices.'

$$\forall x \exists y [E(x, y) \vee E(y, x)]$$

Examples

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- 'The graph is undirected.' (i.e., E is symmetric)

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- 'The graph has no isolated vertices.'

$$\forall x \exists y [E(x,y) \vee E(y,x)]$$

- 'Every vertex has outdegree 1.'

Examples

Graphs $\mathfrak{G} = \langle V, E \rangle,\ E \subseteq V \times V$

- 'The graph is undirected.' (i.e., E is symmetric)

$$\forall x \forall y [E(x, y) \to E(y, x)]$$

- 'The graph has no isolated vertices.'

$$\forall x \exists y [E(x, y) \vee E(y, x)]$$

- 'Every vertex has outdegree 1.'

$$\forall x \exists y [E(x, y) \wedge \forall z [E(x, z) \to z = y]]$$

Normal Forms

Prenex normal form

$$Q_0 x_0 \cdots Q_n x_n \psi(\bar{x}), \quad \psi \text{ quantifier-free}$$

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Skolem normal form

Eliminate **existential quantifiers**:

replace $\forall \bar{x} \exists y \varphi(\bar{x}, y)$ by $\forall \bar{x} \varphi(\bar{x}, f(\bar{x}))$ (f new symbol).

Example

$$\forall x \exists y \exists z [y > x \land z < x]$$

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$$\forall x \exists y \exists z [y > x \wedge z < x] \qquad \forall x [f(x) > x \wedge g(x) < x]$$

$$\exists x \forall y [y + 1 \neq x]$$

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$\forall x \exists y \exists z [y > x \wedge z < x]$	$\forall x [f(x) > x \wedge g(x) < x]$
$\exists x \forall y [y + 1 \neq x]$	$\forall y [y + 1 \neq c]$

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$\exists x \forall y [y + 1 \neq x]$	$\forall y [y + 1 \neq c]$
$\exists x \forall y \exists z \forall u \exists v [R(x, y, z, u, v)]$	

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$$\forall x \exists y \exists z [y > x \wedge z < x] \qquad \forall x [f(x) > x \wedge g(x) < x]$$

$$\exists x \forall y [y + 1 \neq x] \qquad \forall y [y + 1 \neq c]$$

$$\exists x \forall y \exists z \forall u \exists v [R(x, y, z, u, v)] \qquad \forall y \forall u [R(c, y, f(y), u, g(y, u))]$$

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Eliminate **existential quantifiers**:

replace $\forall \bar{x} \exists y \varphi(\bar{x}, y)$ by $\forall \bar{x} \varphi(\bar{x}, f(\bar{x}))$ (f new symbol).

Theorem

Let φ_s be a Skolemisation of φ. Then φ_s is satisfiable iff φ is satisfiable.

Theorem of Herbrand

Theorem of Herbrand

A formula $\exists \bar{x} \varphi(\bar{x})$ is valid if, and only if, there are terms $\bar{t}_0, \ldots, \bar{t}_n$ such that the disjunction $\bigvee_{i \leq n} \varphi(\bar{t}_i)$ is valid.

Corollary

A formula $\forall \bar{x} \varphi(\bar{x})$ is unsatisfiable if, and only if, there are terms $\bar{t}_0, \ldots, \bar{t}_n$ such that the conjunction $\bigwedge_{i \leq n} \varphi(\bar{t}_i)$ is unsatisfiable.

Resolution

Substitution

Definition

A **substitution** σ is a function that replaces in a formula every free variable by a term (and renames bound variables if necessary).
Instead of $\sigma(\varphi)$ we also write $\varphi[x \mapsto s, y \mapsto t]$ if $\sigma(x) = s$ and $\sigma(y) = t$.

Examples

$$
\begin{aligned}
(x = f(y))[x \mapsto g(x),\ y \mapsto c] &\quad = \quad g(x) = f(c) \\
\exists z(x = z + z)[x \mapsto z] &\quad = \quad \exists u(z = u + u)
\end{aligned}
$$

Unification

Definition

A **unifier** of two terms $s(\bar{x})$ and $t(\bar{x})$ is a pair of substitutions σ, τ such that $\sigma(s) = \tau(t)$.
A unifier σ, τ is **most general** if every other unifier σ', τ' can be written as $\sigma' = \rho \circ \sigma$ and $\tau' = \upsilon \circ \tau$, for some ρ, υ.

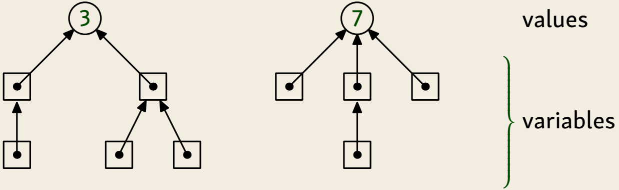
Examples

$s = f(x, g(x))$	$t = f(c, x)$	$x \mapsto c$	$x \mapsto g(c)$
$s = f(x, g(x))$	$t = f(x, y)$	$x \mapsto x$	$x \mapsto x$
			$y \mapsto g(x)$
		$x \mapsto g(x)$	$x \mapsto g(x)$
			$y \mapsto g(g(x))$
$s = f(x)$	$t = g(x)$	unification not possible	

Unification Algorithm

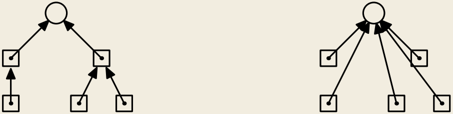
```
unify(s, t)
    if s is a variable x then
        if x already has some value u then
            unify(u, t)
        else
            set x to t
    else if t is a variable x then
        if x already has some value v then
            unify(s, v)
        else
            set x to s
    else s = f(ū) and t = g(v̄)
        if f = g then
            forall i    unify(u_i, v_i)
        else
            fail
```

Union-Find-Algorithm



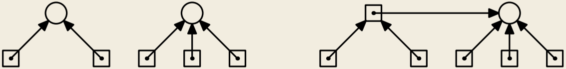
find : *variable* → *value*

- follows pointers to the root and creates shortcuts



union : (*variable* × *variable*) → *unit*

- links roots by a pointer



Clauses

Definitions

- **literal** $R(\bar{t})$ or $\neg R(\bar{t})$
- **clause** set of literals $\{P(\bar{s}), R(\bar{t}), \neg S(\bar{u})\}$

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Example

CNF $\varphi := \forall x \forall y \big[R(x,y) \vee \neg R(x, f(x)) \big] \wedge \forall y \big[\neg R(f(y), y) \vee P(y) \big]$

(no existential quantifiers)

clauses $\big\{ R(x,y)\,, \neg R(x, f(x)) \big\},\ \big\{ \neg R(f(y), y)\,, P(y) \big\}$

Resolution

Resolution Step

Consider two clauses

$$\begin{aligned} C &= \{P(\bar{s}), R_0(\bar{t}_0), \dots, R_m(\bar{t}_m)\} \\ C' &= \{\neg P(\bar{s}'), S_0(\bar{u}_0), \dots, S_n(\bar{u}_n)\} \end{aligned}$$

and let σ, τ be the most general unifier of $\bar{s}$ and $\bar{s}'$. The **resolvent** of C and C' is the clause

$$\{R_0(\sigma(\bar{t}_0)), \dots, R_m(\sigma(\bar{t}_m)), S_0(\tau(\bar{u}_0)), \dots, S_n(\tau(\bar{u}_n))\}.$$

Lemma

Let C be the resolvent of two clauses in Φ. Then

$$\Phi \vDash \Phi \cup \{C\}.$$

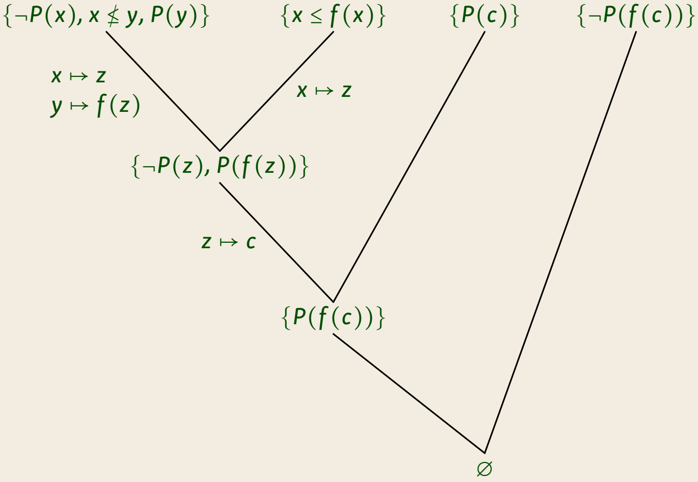
Example

$$\varphi = \forall x \forall y [P(x) \land x \leq y \to P(y)] \land \forall x [x \leq f(x)] \land Pc \land \neg P(f(c))$$

$$\{\neg P(x), x \nleq y, P(y)\} \qquad \{x \leq f(x)\} \qquad \{P(c)\} \qquad \{\neg P(f(c))\}$$

Example

$$\varphi = \forall x \forall y [P(x) \wedge x \leq y \rightarrow P(y)] \wedge \forall x [x \leq f(x)] \wedge Pc \wedge \neg P(f(c))$$



The Resolution Method

Theorem

The resolution method for first-order logic (without equality) is **sound** and **complete.**

Theorem

Satisfiability for first-order logic is **undecidable.**

Satisfiability

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Satisfiability for first-order logic is **undecidable.**

Proof

Turing machine $\mathcal{M} = \langle Q, \Sigma, \Delta, q_0, F_+, F_- \rangle$, non-deterministic

Q	set of states
Σ	tape alphabet
Δ	set of transitions $\langle p, a, b, m, q \rangle \in Q \times \Sigma \times \Sigma \times \{-1, 0, 1\} \times Q$
q_0	initial state
F_+	accepting states
F_-	rejecting states

By adding a counter to $\mathcal{M}$ we may assume that every run of $\mathcal{M}$ terminates.

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Encoding in FO

$S_q(t)$	**state** q at time t
$h(t)$	**head** in field $h(t)$ at time t
$W_a(t, k)$	**letter** a in field k at time t
s	**successor** function $s(n) = n + 1$
0	**zero**

$$\varphi_w := \mathrm{ADM} \wedge \mathrm{INIT} \wedge \mathrm{TRANS} \wedge \mathrm{ACC}$$

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Admissibility formula

$$\mathrm{ADM} := \forall t \bigwedge_{p \neq q} \neg[S_p(t) \wedge S_q(t)] \qquad \text{unique state}$$

$$\wedge \forall t \forall k \bigwedge_{a \neq b} \neg[W_a(t,k) \wedge W_b(t,k)] \qquad \text{unique letter}$$

Proof

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Initialisation formula for input: $a_0 \ldots a_{n-1}$

$$\begin{aligned} \mathrm{INIT} := S_{q_0}(0) & \qquad \text{initial state} \\ \wedge\, h(0) = 0 & \qquad \text{initial head position} \\ \wedge\, \bigwedge_{k<n} W_{a_k}(0, \underline{k}) \wedge \forall k W_{\Box}(0, k+n)] & \qquad \text{initial tape content} \end{aligned}$$

(here $\underline{k} := s(s(\cdots s(0)))$ and $k + n := s^n(k)$)

Acceptance formula

$$\mathrm{ACC} := \forall t \bigwedge_{q \in F_-} \neg S_q(t) \qquad \text{no rejecting states}$$

Proof

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Transition formula

$$\begin{aligned}\mathrm{TRANS} := \forall t \bigvee_{\langle p,a,b,m,q\rangle\in\Delta} \big[& S_p(t) \wedge W_a(t,h(t)) \wedge S_q(s(t)) \wedge \\ & h(s(t)) = h(t) + m \wedge W_b(s(t),h(t))\big] \\ \wedge \forall t \forall k \bigwedge_{a\in\Sigma} \big[& k \neq h(t) \rightarrow [W_a(t,k) \leftrightarrow W_a(s(t),k)]\big]\end{aligned}$$

where

$$y = x + m := \begin{cases} y = s(x) & \text{if } m = 1, \\ y = x & \text{if } m = 0, \\ s(y) = x & \text{if } m = -1. \end{cases}$$

Linear Resolution and Horn Formulae

Horn formulae

A **Horn formulae** is a formula in CNF where each clause contains at most one positive literal.

Theorem

A set of Horn clauses is unsatisfiable if, and only if, one can use linear resolution to derive the empty clause from it.

SLD Resolution

Linear resolution where the clauses are **sequences** instead of sets and we always resolve the **leftmost literal** of the current clause.