IA008: Computational Logic

3. Prolog and Databases

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Prolog

Prolog

Syntax

A **Prolog** program consists of a sequence of statements of the form

$$p(\bar{s}). \quad \text{or} \quad p(\bar{s}) :- q_0(\bar{t}_0), \dots, q_{n-1}(\bar{t}_{n-1}).$$

p, q_i relation symbols, $\bar{s}$, $\bar{t}_i$ tuples of terms.

Semantics

$$p(\bar{s}) :- q_0(\bar{t}_0), \dots, q_{n-1}(\bar{t}_{n-1}).$$

corresponds to the implication

$$\forall \bar{x}\big[p(\bar{s}) \leftarrow q_0(\bar{t}_0) \wedge \dots \wedge q_{n-1}(\bar{t}_{n-1})\big]$$

where $\bar{x}$ are the variables appearing in the formula.

Example

father_of(peter, sam).

father_of(peter, tina).

mother_of(sara, john).

parent_of(X, Y) :– father_of(X, Y).

parent_of(X, Y) :– mother_of(X, Y).

sibling_of(X, Y) :– parent_of(Z, X), parent_of(Z, Y).

ancestor_of(X, Y) :– father_of(X, Z), ancestor_of(Z, Y).

Interpreter

On input

$$p_0(\bar{s}_0), \dots, p_{n-1}(\bar{s}_{n-1}).$$

the program finds all values for the variables satisfying the conjunction

$$p_0(\bar{s}_0) \wedge \cdots \wedge p_{n-1}(\bar{s}_{n-1}).$$

Example

```
?- sibling_of(sam, tina).
Yes

?- sibling_of(X, Y).
X = sam, Y = tina
```

Execution

Input

- program Π (set of Horn formulae
  $\forall \bar{x}[P(\bar{s}) \leftarrow Q_0(\bar{t}_0) \wedge \cdots \wedge Q_{n-1}(\bar{t}_{n-1})]$)
- goal $\varphi(\bar{x}) := R_0(\bar{u}_0) \wedge \cdots \wedge R_{m-1}(\bar{u}_{m-1})$

Evaluation strategy

Use resolution to check for which values of $\bar{x}$ the union $\Pi \cup \{\neg\varphi(\bar{x})\}$ is unsatisfiable.

Remark

As we are dealing with a set of Horn formulae, we can use **linear resolution.** The variant used by Prolog-interpreters is called **SLD-resolution.**

SLD-resolution

- Current goal: $\neg\psi_0 \vee \cdots \vee \neg\psi_{n-1}$
- If $n = 0$, stop.
- Otherwise, find a formula $\psi \leftarrow \vartheta_0 \wedge \cdots \wedge \vartheta_{m-1}$ from Π such that ψ_0 and ψ can be unified.
- If no such formula exists, backtrack.
- Otherwise, resolve them to produce the new goal

$$\tau(\neg\vartheta_0) \vee \cdots \vee \tau(\neg\vartheta_{m-1}) \vee \sigma(\neg\psi_1) \vee \cdots \vee \sigma(\neg\psi_{n-1})\,.$$

(σ, τ is the most general unifier of ψ_0 and ψ.)

Implementation

Use a stack machine that keeps the current goal on the stack.
($\rightarrow$ **Warren Abstract Machine**)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal $\neg$sibling_of(tina, sam)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input $\text{sibling\_of}(\text{tina}, \text{sam})$

goal $\neg\text{sibling\_of}(\text{tina}, \text{sam})$

unify with $\text{sibling\_of}(X, Y) \leftarrow \text{parent\_of}(Z, X) \wedge \text{parent\_of}(Z, Y)$

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal $\neg\text{sibling\_of}(\text{tina}, \text{sam})$

unify with $\text{sibling\_of}(X, Y) \leftarrow \text{parent\_of}(Z, X) \wedge \text{parent\_of}(Z, Y)$

unifier $X = \text{tina}, \; Y = \text{sam}$

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal	$\neg$sibling_of(tina, sam)
unify with	sibling_of(X, Y) $\leftarrow$ parent_of(Z, X) $\wedge$ parent_of(Z, Y)
unifier	X = tina, Y = sam
new goal	$\neg$parent_of(Z, tina), $\neg$parent_of(Z, sam)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal $\neg$parent_of(Z, tina), $\neg$parent_of(Z, sam)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal $\neg$parent_of(Z, tina), $\neg$parent_of(Z, sam)

unify with parent_of(X, Y) $\leftarrow$ mother_of(X, Y)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal	$\neg\text{parent\_of}(Z, \text{tina}),\ \neg\text{parent\_of}(Z, \text{sam})$
unify with	$\text{parent\_of}(X, Y) \leftarrow \text{mother\_of}(X, Y)$
unifier	$X = Z,\ Y = \text{tina}$

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal	$\neg$parent_of(Z, tina), $\neg$parent_of(Z, sam)
unify with	parent_of(X, Y) $\leftarrow$ mother_of(X, Y)
unifier	$X = Z$, $Y =$ tina
new goal	$\neg$mother_of(Z, tina), $\neg$parent_of(Z, sam)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal $\neg$mother_of(Z, tina), $\neg$parent_of(Z, sam)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal	¬mother_of(Z, tina), ¬parent_of(Z, sam)
unify with	mother_of(sara, john)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal $\neg$mother_of(Z, tina), $\neg$parent_of(Z, sam)

unify with mother_of(sara, john)

fails

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal $\neg$mother_of(Z, tina), $\neg$parent_of(Z, sam)

unify with mother_of(sara, john)

fails

backtrack to $\neg$parent_of(Z, tina), $\neg$parent_of(Z, sam)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal $\neg$parent_of(Z, tina), $\neg$parent_of(Z, sam)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal $\neg\text{parent\_of}(Z, \text{tina}),\ \neg\text{parent\_of}(Z, \text{sam})$

unify with $\text{parent\_of}(X, Y) \leftarrow \text{father\_of}(X, Y)$

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal	$\neg\text{parent\_of}(Z, \text{tina}),\ \neg\text{parent\_of}(Z, \text{sam})$
unify with	$\text{parent\_of}(X, Y) \leftarrow \text{father\_of}(X, Y)$
unifier	$X = Z,\ Y = \text{tina}$

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal	$\neg$parent_of(Z, tina), $\neg$parent_of(Z, sam)
unify with	parent_of(X, Y) $\leftarrow$ father_of(X, Y)
unifier	$X = Z,\ Y =$ tina
new goal	$\neg$father_of(Z, tina), $\neg$parent_of(Z, sam)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal $\neg$father_of(Z, tina), $\neg$parent_of(Z, sam)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal $\neg$father_of(Z, tina), $\neg$parent_of(Z, sam)

unify with father_of(peter, sam)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal $\neg$father_of(Z, tina), $\neg$parent_of(Z, sam)

unify with father_of(peter, sam)

fails

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal	¬father_of(Z, tina), ¬parent_of(Z, sam)
unify with	father_of(peter, sam)
fails	
unify with	father_of(peter, tina)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal	$\neg$father_of(Z, tina), $\neg$parent_of(Z, sam)
unify with	father_of(peter, sam)
fails	
unify with	father_of(peter, tina)
unifier	Z = peter

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal	$\neg$father_of(Z, tina), $\neg$parent_of(Z, sam)
unify with	father_of(peter, sam)
fails	
unify with	father_of(peter, tina)
unifier	Z = peter
new goal	$\neg$parent_of(peter, sam)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal ¬parent_of(peter, sam)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal	¬parent_of(peter, sam)
...	...
goal	¬father_of(peter, sam)

Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal	¬parent_of(peter, sam)
...	...
goal	¬father_of(peter, sam)
unify with	father_of(peter, sam)

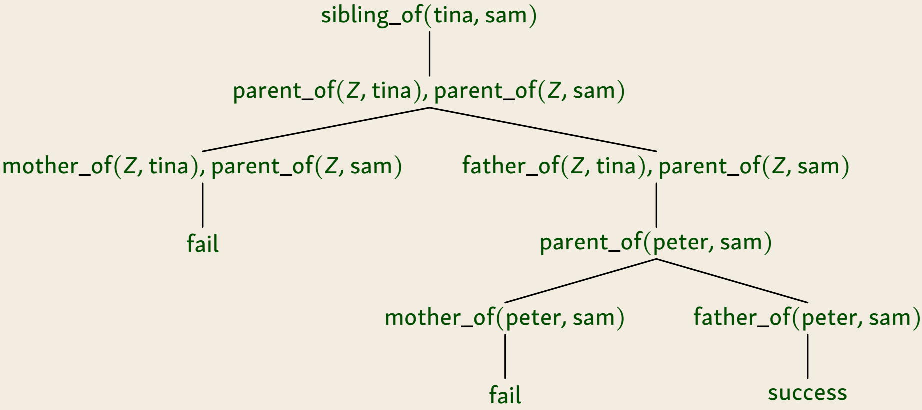
Example

```
father_of(peter, sam).
father_of(peter, tina).
mother_of(sara, john).
parent_of(X, Y) :- mother_of(X, Y).
parent_of(X, Y) :- father_of(X, Y).
sibling_of(X, Y) :- parent_of(Z, X), parent_of(Z, Y).
```

Input sibling_of(tina, sam)

goal	¬parent_of(peter, sam)
...	...
goal	¬father_of(peter, sam)
unify with	father_of(peter, sam)
new goal	empty

Search tree



sibling_of(tina, sam)
parent_of(Z, tina), parent_of(Z, sam)
mother_of(Z, tina), parent_of(Z, sam)
father_of(Z, tina), parent_of(Z, sam)
fail
parent_of(peter, sam)
mother_of(peter, sam)
father_of(peter, sam)
fail
success

Caveats

Prolog-interpreters use a simpler (and **unsound**) form of unification that ignores multiple occurrences of variables. E.g. they happily unify $p(x, f(x))$ with $p(f(y), f(y))$ (equating $x = f(y)$ for the first x and $x = y$ for the second one).

Caveats

Prolog-interpreters use a simpler (and **unsound**) form of unification that ignores multiple occurrences of variables. E.g. they happily unify $p(x, f(x))$ with $p(f(y), f(y))$ (equating $x = f(y)$ for the first x and $x = y$ for the second one).

It is also easy to get infinite loops if you are not careful with the ordering of the rules:

```
edge(c,d).
path(X,Y) :- path(X,Z),edge(Z,Y).
path(X,Y) :- edge(X,Y).
```

produces

```
?- path(X,Y).
path(X,Z), edge(Z,Y).
path(X,U), edge(U,Z), edge(Z,Y).
path(X,V), edge(V,U), edge(U,Z), edge(Z,Y).
...
```

Example: List processing

```
append([], L, L).
append([H|T], L, [H|R]) :- append(T, L, R).

?- append([a,b], [c,d], X).
X = [a,b,c,d]

?- append(X, Y, [a,b,c,d])
X = [], Y = [a,b,c,d]
X = [a], Y = [b,c,d]
X = [a,b], Y = [c,d]
X = [a,b,c], Y = [d]
X = [a,b,c,d], Y = []
```

Example: List processing

```
reverse(Xs, Ys) :- reverse_(Xs, [], Ys).

reverse_([], Ys, Ys).
reverse_([X|Xs], Rs, Ys) :- reverse_(Xs, [X|Rs], Ys).

reverse([a,b,c], X)
reverse_([a,b,c], [], X)
reverse_([b,c], [a], X)
reverse_([c], [b,a], X)
reverse_([], [c,b,a], X)
X = [c,b,a]
```

Example: Natural language recognition

```
sentence(X,R) :- noun(X, Y), verb(Y, R).
sentence(X,R) :- noun(X, Y), verb(Y, Z), noun(Z, R).

noun_phrase(X,             R) :- noun(X, R).
noun_phrase(['a'   | X], R) :- noun(X, R).
noun_phrase(['the' | X], R) :- noun(X, R).

noun(['cat'   | R], R).
noun(['mouse' | R], R).
noun(['dog'   | R], R).

verb(['eats'  | R], R).
verb(['hunts' | R], R).
verb(['plays' | R], R).
```

Cuts

Control backtracking using **cuts**:

$$p \mathrel{:-} q_0, q_1, !, q_2, q_3.$$

When backtracking across a cut $!$, directly jump to the head of the rule and assume it fails. Do not try other rules.

Example

$s \leftarrow p$

$s \leftarrow t$

$p \leftarrow q_1, q_2, !, q_3, q_4$

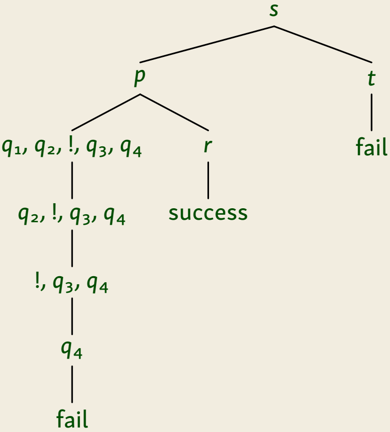
$p \leftarrow r$

r

q_1

q_2

q_3



Negation

Problem

If we allow **negation,** the formulae are no longer **Horn** and SLD-resolution does no longer work.

Possible Solutions

- **Closed World Assumption**
  If we cannot derive p, it is false (**Negation as Failure**).
- **Completed Database**
  $p \leftarrow q_0, \ldots, p \leftarrow q_n$ is interpreted as the stronger statement $p \leftrightarrow q_0 \vee \cdots \vee q_n$.

Examples

Being connected by a path of non-edges:

```
q(X,X).
q(X,Y) :- q(X,Z), not(p(Z,Y)).
```

Implementing negation using cuts:

```
not(A) :- A, !, fail.
not(A).
```

Some cuts can be implemented using negation:

```
p :- a, !, b.          p :- a, b.
p :- c.                p :- not(a), c.
```

Databases

Databases

Definition

A **database** is a set of relations called **tables.**

Example

flight	from	to	price
LH8302	Prague	Frankfurt	240
OA1472	Vienna	Warsaw	300
UA0870	London	Washington	800
...			

Formal Definitions

We treat a database as a structure $\mathfrak{A} = \langle A, R_0, \dots, R_n \rangle$ with

- universe A containing all entries and
- one relation $R_i \subseteq A \times \cdots \times A$ per table.

The **active domain** of a database is the set of elements appearing in some relation.

Example

In the previous table, the active domain contains:

LH8302, OA1472, UA0870, 240, 300, 800,
Prague, Frankfurt, Vienna, Warsaw, London, Washington

Queries

A **query** is a function mapping each database to a relation.

Example

Input: database of direct flights
Output: table of all flight connections possibly including stops

In Prolog: database **flight**, query **connection**.

```
flight('LH8302', 'Prague', 'Frankfurt',  240).
flight('OA1472', 'Vienna', 'Warsaw',     300).
flight('UA0870', 'London', 'Washington', 800).

connection(From, To) :- flight(X, From, To, Y).
connection(From, To) :-
  flight(X, From, T, Y), connection(T, To).
```

Relational Algebra

Syntax

- basic relations
- boolean operations $\cap$, $\cup$, $\setminus$, All
- cartesian product $\times$
- selection σ_{ij}
- projection $\pi_{u_0 \dots u_{n-1}}$

Examples

- $\pi_{1,0}(R) = \{\, (b, a) \mid (a, b) \in R \,\}$
- $\pi_{0,3}(\sigma_{1,2}(E \times E)) = \{\, (a, c) \mid (a, b), (b, c) \in E \,\}$

Relational Algebra

Syntax

- basic relations
- boolean operations $\cap$, $\cup$, $\setminus$, All
- cartesian product $\times$
- selection σ_{ij}
- projection $\pi_{u_0 \dots u_{n-1}}$

Examples

- $\pi_{1,0}(R) = \{ (b, a) \mid (a, b) \in R \}$
- $\pi_{0,3}(\sigma_{1,2}(E \times E)) = \{ (a, c) \mid (a, b), (b, c) \in E \}$

Join

$$R \bowtie_{ij} S := \sigma_{ij}(R \times S)$$

Expressive Power

Theorem

Relational Algebra = First-Order Logic

Expressive Power

Theorem

Relational Algebra $=$ First-Order Logic

Proof

$(\leq)$ $s \mapsto s^*$ such that $s = \{\, \bar{a} \mid \mathfrak{A} \models s^*(\bar{a}) \,\}$

Expressive Power

Theorem

Relational Algebra = First-Order Logic

Proof

$(\leq)$ $s \mapsto s^*$ such that $s = \{\, \bar{a} \mid \mathfrak{A} \models s^*(\bar{a}) \,\}$

$$\begin{aligned}
R^* &:= R(x_0, \dots, x_{n-1}) \\
(s \cap t)^* &:= s^* \wedge t^* \\
(s \cup t)^* &:= s^* \vee t^* \\
(s \setminus t)^* &:= s^* \wedge \neg t^* \\
\mathrm{All}^* &:= \mathrm{true} \\
(s \times t)^* &:= s^*(x_0, \dots, x_{m-1}) \wedge t^*(x_m, \dots, x_{m+n-1}) \\
\sigma_{ij}(s)^* &:= s^* \wedge x_i = x_j \\
\pi_{u_0,\dots,u_{n-1}}(s)^* &:= \exists \bar{y} \Big[s^*(\bar{y}) \wedge \bigwedge_{i<n} x_i = y_{u_i} \Big]
\end{aligned}$$

Expressive Power

Theorem

Relational Algebra = First-Order Logic

Proof

$(\geq)$ $\varphi \mapsto \varphi^*$ such that $\varphi^* = \{\, \bar{a} \mid \mathfrak{A} \vDash \varphi(\bar{a}) \,\}$

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$(\geq)$ $\varphi \mapsto \varphi^*$ such that $\varphi^* = \{\, \bar{a} \mid \mathfrak{A} \vDash \varphi(\bar{a}) \,\}$

$$\begin{aligned}
R(x_{u_0}, \dots, x_{u_{n-1}})^* &:= \pi_{0,\dots,m-1}(\sigma_{u_0,m+0} \cdots \sigma_{u_{n-1},m+n-1} \\
&\qquad\qquad\qquad\qquad (\mathrm{All} \times \cdots \times \mathrm{All} \times R)) \\
(x_i = x_j)^* &:= \sigma_{ij}(\mathrm{All} \times \cdots \times \mathrm{All}) \\
(\varphi \wedge \psi)^* &:= \varphi^* \cap \psi^* \\
(\varphi \vee \psi)^* &:= \varphi^* \cup \psi^* \\
(\neg\varphi)^* &:= \mathrm{All} \times \cdots \times \mathrm{All} \setminus \varphi^* \\
(\exists x_i \varphi)^* &:= \pi_{0,\dots,i-1,n,i+1,\dots,n-1}(\varphi^* \times \mathrm{All})
\end{aligned}$$

Query Evaluation

Conjunctive query

$$\varphi(\bar{x}) = \exists \bar{y}\big[R_0(\bar{z}_0) \wedge \cdots \wedge R_{n-1}(\bar{z}_{n-1})\big] \qquad \text{for } \bar{z}_0, \dots, \bar{z}_{n-1} \subseteq \bar{x}\bar{y}$$

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Relational Algebra

$$\pi_{\bar{u}}[R_0 \times \cdots \times R_{n-1}]$$

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Relational Algebra

$$\pi_{\bar{u}}[R_0 \times \cdots \times R_{n-1}]$$

Query Optimisation

$$\pi_{\bar{u}}[R_0 \bowtie_{ij} \cdots \bowtie_{kl} R_{n-1}]$$ (works if φ is 'tree-shaped')

Constraint Satisfaction Problem (CSP)

Given structures $\mathfrak{A}$, $\mathfrak{B}$, does there exists a homomorphism $\mathfrak{A} \to \mathfrak{B}$?

Constraint Satisfaction Problem (CSP)

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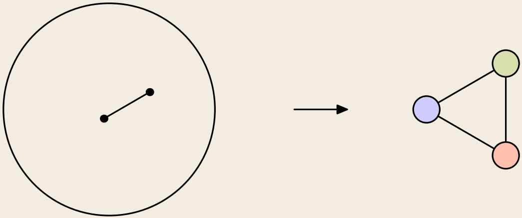
Examples

- 3-Colourability



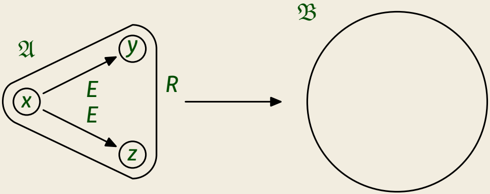
- Sodoku $\langle 9 \times 9, \neq \rangle \to \langle 9, \neq \rangle$

Constraint Satisfaction Problem (CSP)

Given structures $\mathfrak{A}$, $\mathfrak{B}$, does there exists a homomorphism $\mathfrak{A} \to \mathfrak{B}$?

Example

$$\varphi(x) = \exists y \exists z [E(x,y) \wedge E(x,z) \wedge R(x,y,z)]$$



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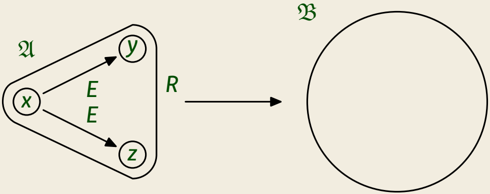


Complexity

In general the problem is **NP-complete,** but there are subclasses where it is in **P.**

Datalog

Datalog

Simplified version of Prolog developped in database theory:

- no function symbols,
- no cut, no negation, etc.

A **datalog program** for a database $\mathfrak{A} = \langle A, R_0, \dots, R_n \rangle$ is a set of Horn formulae

$$
\begin{aligned}
p_0(\bar{X}) &\leftarrow q_{0,0}(\bar{X}, \bar{Y}) \wedge \cdots \wedge q_{0,m_0}(\bar{X}, \bar{Y}) \\
&\vdots \\
p_n(\bar{X}) &\leftarrow q_{n,0}(\bar{X}, \bar{Y}) \wedge \cdots \wedge q_{n,m_n}(\bar{X}, \bar{Y})
\end{aligned}
$$

where $p_0, \dots, p_n$ are **new** relation symbols and the q_{ij} are either relation symbols from $\mathfrak{A}$, possibly negated, or one of the new symbols p_k (not negated).

Datalog queries

The **query** defined by a datalog program

$$
\begin{aligned}
p_0(\bar{X}) &\leftarrow q_{0,0}(\bar{X},\bar{Y}) \wedge \cdots \wedge q_{0,m_0}(\bar{X},\bar{Y}) \\
&\vdots \\
p_n(\bar{X}) &\leftarrow q_{n,0}(\bar{X},\bar{Y}) \wedge \cdots \wedge q_{n,m_n}(\bar{X},\bar{Y})
\end{aligned}
$$

maps a database $\mathfrak{A}$ to the relations $p_0, \ldots, p_n$ defined by these formulae.

Evaluation strategy

- Start with empty relations $p_0 = \emptyset, \ldots, p_n = \emptyset$.
- Apply each rule to add new tuples to the relations.
- Repeat until no new tuples are generated.

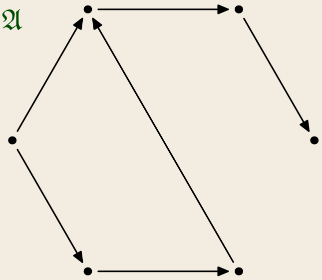
Note

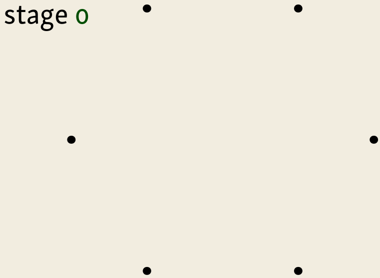
The relations computed in this way satisfy the **Completed Database** assumption.

Example

$$\mathrm{path}(X, Y) \leftarrow \mathrm{edge}(X, Y)$$
$$\mathrm{path}(X, Y) \leftarrow \mathrm{path}(X, Z) \wedge \mathrm{path}(Z, Y)$$

$\mathfrak{A}$





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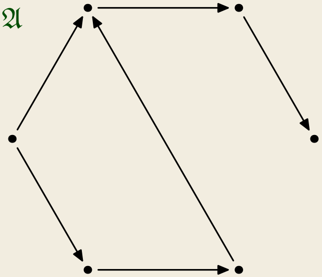
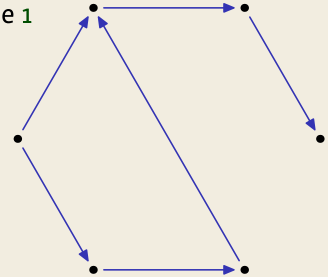


stage 1



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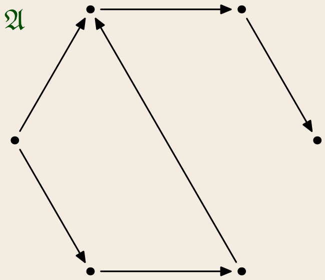

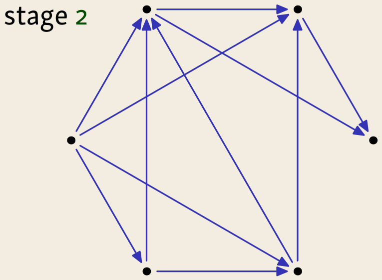

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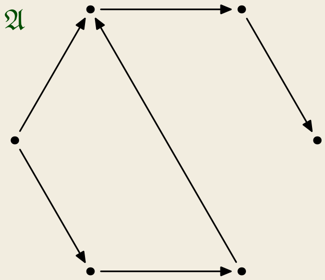

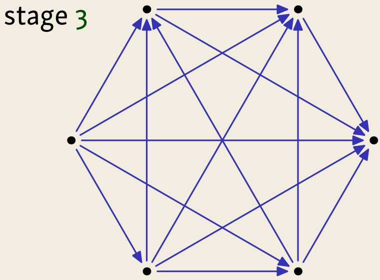

Example: Arithmetic

$$
\begin{aligned}
\mathrm{Add}(x, y, z) &\leftarrow y = 0 \land x = z \\
\mathrm{Add}(x, y, z) &\leftarrow E(y', y) \land E(z', z) \land \mathrm{Add}(x, y', z') \\
\mathrm{Mul}(x, y, z) &\leftarrow y = 0 \land z = 0 \\
\mathrm{Mul}(x, y, z) &\leftarrow E(y', y) \land \mathrm{Add}(x, z', z) \land \mathrm{Mul}(x, y', z')
\end{aligned}
$$

stage 0 $\varnothing$

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stage 0	$\varnothing$
stage 1	$(k, 0, k)$

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\end{aligned}$$

stage 0	$\varnothing$
stage 1	$(k, 0, k)$
stage 2	$(k, 0, k), (k, 1, k+1)$

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stage 0	$\emptyset$
stage 1	$(k, 0, k)$
stage 2	$(k, 0, k), (k, 1, k+1)$
stage 3	$(k, 0, k), (k, 1, k+1), (k, 2, k+2)$
	$\ldots$
stage n	$(k, 0, k), (k, 1, k+1), \ldots, (k, n-1, k+n-1)$
	$\ldots$

Complexity

Theorem

For databases $\mathfrak{A} = \langle A, \bar{R}, \leq \rangle$ equipped with a linear order $\leq$, a query Q can be expressed as a Datalog program if, and only if, it can be evaluated in **polynomial type.**