IA008: Computational Logic

5. Ehrenfeucht-Fraïssé Games

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Quantifier rank

Definition

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\begin{aligned}
\mathrm{qr}(s = t) &:= 0\,, & \mathrm{qr}(\varphi \wedge \psi) &:= \max\{\mathrm{qr}(\varphi), \mathrm{qr}(\psi)\}\,,\\
\mathrm{qr}(R\bar{t}) &:= 0\,, & \mathrm{qr}(\varphi \vee \psi) &:= \max\{\mathrm{qr}(\varphi), \mathrm{qr}(\psi)\}\,,\\
\mathrm{qr}(\exists x\varphi) &:= 1 + \mathrm{qr}(\varphi)\,, & \mathrm{qr}(\neg\varphi) &:= \mathrm{qr}(\varphi)\,,\\
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Example

$$\mathrm{qr}\big(\forall x \exists y R(x, y)\big) = 2\,,$$

$$\mathrm{qr}\big(\forall x[P(x) \vee Q(x)] \wedge \forall z[\exists y R(y, z) \vee \exists y R(z, y)]\big) = 2\,.$$

Quantifier rank

Lemma

Up to logical equivalence, there are only finitely many formulae of quantifier rank at most m with free variables $\bar{x}$.
(For a fixed signature Σ that is finite and relational.)

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Back-and-forth equivalence

m-equivalence

$$\mathfrak{A}, \bar{a} \equiv_m \mathfrak{B}, \bar{b} \quad :\text{iff} \quad \mathfrak{A} \models \varphi(\bar{a}) \;\Leftrightarrow\; \mathfrak{B} \models \varphi(\bar{b}),$$

for all $\varphi(\bar{x})$ with $\mathrm{qr}(\varphi) \leq m$.

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Lemma

$$\mathfrak{A}, \bar{a} \equiv_{m+1} \mathfrak{B}, \bar{b}$$

if, and only if,

- for all $c \in A$, exists $d \in B$ with $\mathfrak{A}, \bar{a}c \equiv_m \mathfrak{B}, \bar{b}d$ and
- for all $d \in B$, exists $c \in A$ with $\mathfrak{A}, \bar{a}c \equiv_m \mathfrak{B}, \bar{b}d$.

('back-and-forth conditions')

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Suppose $\mathfrak{A} \models \exists x \varphi(\bar{a}, x)$ with $\mathrm{qr}(\varphi) \leq m$.

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$\Rightarrow \mathfrak{A}, \bar{a}c \equiv_m \mathfrak{B}, \bar{b}d$

Ehrenfeucht-Fraïssé Games

Game $\mathcal{G}_m(\mathfrak{A}, \bar{a}; \mathfrak{B}, \bar{b})$

Players: **Spoiler** and **Duplicator**

m **rounds:**

- Spoiler picks an element of one structure.
- Duplicator picks an element of the other structure.

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Theorem

$\mathfrak{A}, \bar{a} \equiv_m \mathfrak{B}, \bar{b}$ if, and only if, Duplicator wins $\mathcal{G}_m(\mathfrak{A}, \bar{a}; \mathfrak{B}, \bar{b})$

Ehrenfeucht-Fraïssé Games

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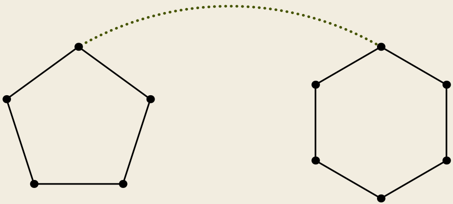

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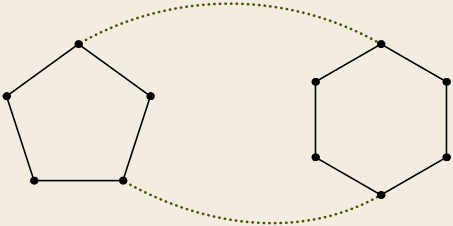

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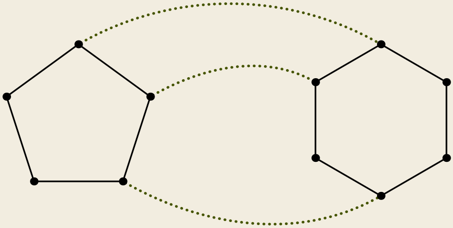

Back-and-forth equivalence

Example linear orders

$$\langle A, \leq \rangle \equiv_m \langle B, \leq \rangle \quad \text{iff} \quad |A| = |B| \quad \text{or} \quad |A|, |B| \geq 2^m - 1\,.$$

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For $2^{m-1} \leq k < 2^m$, construct $\varphi_k(x, y)$ with $\mathrm{qr}(\varphi_k) = m$ stating that $x < y$ and there are at least k elements between x and y.

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$$\varphi_0(x, y) := x < y \,,$$
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By inductive hypothesis, Duplicator can then continue the game for $m - 1$ rounds.

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Corollary

There does not exist an FO-formula φ such that

$$\langle A, \leq \rangle \vDash \varphi \quad \text{iff} \quad |A| \text{ is even,}$$

for all finite linear orders.

Words

Word structures

We can represent $u = a_0 \dots a_{n-1} \in \Sigma^*$ as a structure

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Corollary

For $L \subseteq \Sigma^*$ FO-definable, there exists $n \in \mathbb{N}$ such that

$$uv^n w \in L \Leftrightarrow uv^{n+1} w \in L\,, \quad \text{for all } u, v, w \in \Sigma^*.$$

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- all correctly parenthesised expressions over the alphabet $(\ )\ x$

Monadic Second-Order Logic

Syntax

- element variables: $x, y, z, \dots$
- set variables: $X, Y, Z, \dots$
- atomic formulae: $R(\bar{x})$, $x = y$, $x \in X$
- boolean operations: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
- quantifiers: $\exists x, \forall x, \exists X, \forall X$

Example

- "The set X is empty."

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- "There exists a path from x to y."

  $\forall Z\big[x \in Z \wedge \forall u \forall v[u \in Z \wedge E(u, v) \rightarrow v \in Z] \rightarrow y \in Z\big]$

Back-and-Forth Equivalence

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$$\mathfrak{A}, \bar{P}, \bar{a} \equiv_m^{\mathrm{MSO}} \mathfrak{B}, \bar{Q}, \bar{b} \quad :\text{iff} \quad \mathfrak{A} \vDash \varphi(\bar{P}, \bar{a}) \Leftrightarrow \mathfrak{B} \vDash \varphi(\bar{Q}, \bar{b})$$

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$$[\varepsilon]_m,\ [a_0]_m,\ [a_0 a_1]_m,\ \dots,\ [a_0 a_1 \cdots a_{n-1}]_m\,.$$

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Corollary

φ is satisfiable (by a finite word) if, and only if, $\mathcal{A}_\varphi$ accepts some word.

Automata

Given $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ construct $\varphi_{\mathcal{A}}$.

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$\Rightarrow a^k b^i \in L$ Contradiction.

The Theorem of Gaifman

Gaifman graph

$\mathcal{G}(\mathfrak{A}) := \langle A, E \rangle$ where $E := \{ \langle c_i, c_j \rangle \mid \bar{c} \in R, \ c_i \neq c_j \}$

$d(x, y)$ distance in $\mathcal{G}(\mathfrak{A})$

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Basic local sentence

$$\varphi = \exists x_0 \dots x_{n-1}\Big[\bigwedge_{i \neq j} d(x_i, x_j) \geq 2r \wedge \bigwedge_{i<n} \psi^{(r)}(x_i)\Big]$$

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Theorem

Every FO-formula $\varphi(\bar{x})$ is equivalent to a boolean combination of

- basic local sentences and
- formulae of the form $\psi^{(r)}(x_i)$.

Examples

Connectivity is not first-order definable.

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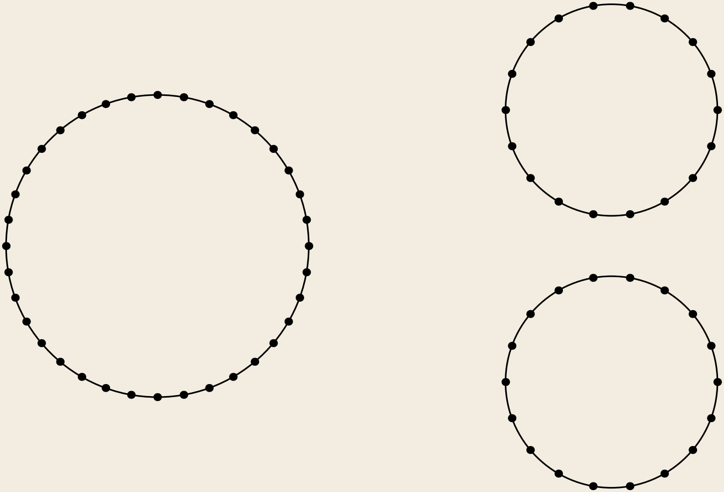

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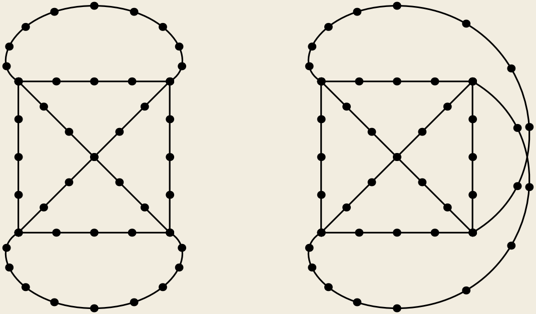

Proof

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same basic local sentences up to qr $h(r)$.

$$N(\bar{a}, r) := \{\, c \mid d(c, a_i) < r \text{ for some } i \,\}$$

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Claim $\quad N(\bar{a}, 7^r), \bar{a} \equiv_{g(r)} N(\bar{b}, 7^r), \bar{b} \quad \Rightarrow \quad \mathfrak{A}, \bar{a} \equiv_r \mathfrak{B}, \bar{b}$

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Using this claim, we can proof the theorem as follows.
Let $r := \mathrm{qr}(\varphi)$. It is sufficient to show that

$$\varphi(\bar{x}) \equiv \bigvee \{\, \chi_{\mathfrak{A},\bar{a}}(\bar{x}) \mid \mathfrak{A} \vDash \varphi(\bar{a}) \,\},$$

where

$$\chi_{\mathfrak{A},\bar{a}}(\bar{x}) := \bigwedge \Theta_{\mathfrak{A}} \wedge \bigwedge \Theta'_{\mathfrak{A},\bar{a}}(\bar{x}),$$

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Claim There exists $\psi_{\bar{a},r,m}(\bar{x})$ such that

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Set

$$\psi_{\bar{a},r,m} := \bigwedge \Theta$$

where

$$\Theta := \left\{\, \vartheta^{(r)}(\bar{x}) \;\middle|\; \mathrm{qr}(\vartheta) \leq m\,,\; N(\bar{a}, r) \models \vartheta(\bar{a}) \,\right\}$$

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Claim $\quad N(\bar{a}, 7^r), \bar{a} \equiv_{g(r)} N(\bar{b}, 7^r), \bar{b} \quad \Rightarrow \quad \mathfrak{A}, \bar{a} \equiv_r \mathfrak{B}, \bar{b}$

Induction on r

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Induction on r

$(r+1)$ Fix $c \in A$.

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Then $N(c, 7^r) \subseteq N(\bar{a}, 7^{r+1})$ and

$$\begin{aligned}
& N(\bar{a}, 7^{r+1}), \bar{a} \equiv_{g(r)+k+m+1} N(\bar{b}, 7^{r+1}), \bar{b} \\
\Rightarrow\ & N(\bar{a}, 7^{r+1}), \bar{a}c \equiv_{g(r)+m} N(\bar{b}, 7^{r+1}), \bar{b}d \quad \text{for some } d \in N(\bar{b}, 2 \cdot 7^r), \\
\Rightarrow\ & N(\bar{a}c, 7^r), \bar{a}c \equiv_{g(r)} N(\bar{b}d, 7^r), \bar{b}d .
\end{aligned}$$

$g(r+1) \geq g(r) + k + m + 1$ where k, m are the quantifier-ranks of the formulae defining $N(\bar{a}, 2 \cdot 7^r)$ and $N(\bar{a}c, 7^r)$.

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$$\vartheta_k(\bar{x}) := \bigwedge_{i \neq j} d(x_i, x_j) \geq 4 \cdot 7^r \wedge \bigwedge_i [N(x_i, 7^r) \equiv_{g(r)} N(c, 7^r)] .$$

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Let k be maximal such that $N(\bar{a}, 2 \cdot 7^r)$ contains k elements $\bar{c}'$ with

$$N(\bar{a}, 7^{r+1}) \vDash \vartheta_k(\bar{c}') .$$

(Note that $k \leq |\bar{a}|$.)

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Case 2 a $\mathfrak{B} \vDash \exists \bar{x} \vartheta_{k+1}(\bar{x})$ (c is far away from the $\bar{a}$)

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Case 2 a $\mathfrak{B} \vDash \exists \bar{x} \vartheta_{k+1}(\bar{x})$ (c is far away from the $\bar{a}$)

Then there is some $d \notin N(\bar{b}, 2 \cdot 7^r)$ with

$$N(d, 7^r) \equiv_{g(r)} N(c, 7^r) .$$

Proof

Case 2 $c \notin N(\bar{a}, 2 \cdot 7^r)$

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Case 2 b $\mathfrak{B} \vDash \neg \exists \bar{x} \vartheta_{k+1}(\bar{x})$ (c is at medium distance from the $\bar{a}$)

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Case 2 $c \notin N(\bar{a}, 2 \cdot 7^r)$

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Case 2 b $\mathfrak{B} \models \neg\exists\bar{x}\vartheta_{k+1}(\bar{x})$ (c is at medium distance from the $\bar{a}$)

$\Rightarrow \mathfrak{A} \models \neg\exists\bar{x}\vartheta_{k+1}(\bar{x})$

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$\Rightarrow \mathfrak{A} \vDash \neg\exists\bar{x}\vartheta_{k+1}(\bar{x})$

$\Rightarrow c \in N(\bar{a}, 7^{r+1})$ satisfies $2 \cdot 7^r \leq d(c, a_i) < 6 \cdot 7^r$

$\Rightarrow$ There is some $d \in N(\bar{b}, 7^{r+1})$ such that

$$2 \cdot 7^r \leq d(d, b_i) < 6 \cdot 7^r \quad \text{and} \quad N(d, 7^r) \equiv_{g(r)} N(c, 7^r) .$$