

Exercise 1 Show that the following formulae are valid using tableau proofs.

- (a) $\psi \rightarrow ((\varphi \wedge \psi) \vee \psi)$
- (b) $((\varphi \wedge \psi) \vee \psi) \rightarrow \psi$
- (c) $\varphi \rightarrow \exists x\varphi$
- (d) $\forall x\varphi \rightarrow \varphi$
- (e) $\forall x(\varphi \wedge \psi) \rightarrow \forall x\varphi \wedge \forall x\psi$
- (f) $\exists x(\varphi \vee \psi) \rightarrow (\exists x\varphi \vee \exists x\psi)$
- (g) $\varphi \wedge \psi \rightarrow \psi \wedge \varphi$
- (h) $(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi)$
- (i) $\varphi \rightarrow \neg\neg\varphi$
- (j) $(\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \vartheta) \rightarrow (\varphi \rightarrow \psi \wedge \vartheta)$
- (k) $(\varphi \rightarrow \psi \wedge \vartheta) \rightarrow (\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \vartheta)$
- (l) $\neg\neg\varphi \rightarrow \varphi$
- (m) $\varphi \vee \neg\varphi$
- (n) $\neg(\neg\varphi \wedge \neg\psi) \rightarrow (\varphi \vee \psi)$
- (o) $\forall xR(x, x) \rightarrow \forall x\exists yR(f(x), y)$
- (p) $(\exists x\varphi \vee \exists x\psi) \rightarrow \exists x(\varphi \vee \psi)$
- (q) $\forall x\varphi \wedge \forall x\psi \rightarrow \forall x(\varphi \wedge \psi)$
- (r) $\forall x\forall y[\varphi(x) \leftrightarrow \varphi(y)] \wedge \exists x\varphi(x) \rightarrow \forall x\varphi(x)$

Exercise 2 Prove that the formulae from Exercise 1 are valid using Natural Deduction.

Exercise 3 Find all consistent sets for the following sets of rules.

- (a) $\frac{\quad}{\alpha} \quad \frac{\alpha : \beta}{\delta} \quad \frac{\alpha : \gamma}{\delta}$
- (b) $\frac{\quad}{\alpha} \quad \frac{\alpha : \beta \gamma}{\beta}$
- (c) $\frac{\quad}{\alpha} \quad \frac{\alpha \beta}{\gamma} \quad \frac{\alpha : \gamma}{\beta}$

Exercise 4 For each of the following subsets $\Phi \subseteq \mathcal{P}(\{\alpha, \beta\})$, find a set of rules R such that Φ is the set of all consistent sets for R .

(a) $\{\emptyset, \{\alpha\}, \{\alpha, \beta\}\}$

(b) $\{\{\alpha\}, \{\beta\}, \{\alpha, \beta\}\}$

(c) $\{\emptyset, \{\alpha, \beta\}\}$

(d) $\{\{\alpha\}, \{\alpha, \beta\}\}$

Exercise 5 (optional) Derive the following additional rules from the basic ones of the Natural Deduction calculus (that is, combine the basic rules to obtain the ones below).

$$\frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi} \quad \frac{\Gamma \vdash \varphi}{\Gamma, \Delta \vdash \varphi} \quad \frac{\Gamma, \neg\varphi \vdash \neg\psi}{\Gamma \vdash \psi \rightarrow \varphi}$$