Exercise 1 We consider structures with the empty signature. Use a back-and-forth argument to show that

$$\mathfrak{A} \equiv_m \mathfrak{B} \quad \text{iff} \quad |A| = |B| \quad \text{or} \quad |A|, |B| \geq m\,.$$

Exercise 2 Prove that the set of all complete finite binary trees whose number of vertices is divisible by 3 is not first-order definable.

Exercise 3 Which of the following languages (over $\{a, b\}$) are definable in first-order logic? Which of them are definable in monadic second-order logic?

(a) $a^*(bb)^*$

(b) $(ab)^*$

(c) $\{ a^{n^2} \mid n \in \mathbb{N} \}$

(d) $\{ a^n b^m a^n \mid m, n \in \mathbb{N} \}$

(e) The set of all palindromes.

Exercise 4 Which of the following graph classes are first-order definable? (We assume that all graphs are finite and undirected.)

(a) graphs of degree at most 3

(b) paths

(c) trees

(d) graphs of diameter at most 3

(e) graphs of even diameter

(f) graphs with a Hamiltonian cycle

Exercise 5 (optional) Prove that the set of even numbers is not first-order definable in $\langle \mathbb{Z}, \leq \rangle$.