Sada 2

March 1, 2025

1. Compute explicitely $\chi(\Delta^n)$, where $\Delta^n = \{S \mid S \subseteq \{0, \ldots, n\}\}$.
2. Given two simplicial complexes A, B with disjoint vertex sets $V(A)$, $V(B)$, their *join* $A * B$ is defined by
$$A * B = \{\sigma \cup \tau \mid \sigma \in A, \tau \in B\}.$$
Derive a formula for computing $\chi(A * B)$ in terms of $\chi(A)$, $\chi(B)$.
3. Given two simplicial complexes A, B, derive a formula for $\chi(A \cup B)$ in terms of $\chi(A)$, $\chi(B)$, $\chi(A \cap B)$.
4. Find values of n, d such that for an n-tuple of points in general position in $\mathbb{R}^d$, the alpha complex and Vietoris-Rips complex coincide.
5. Find values of n, d such that for an n-tuple of points in general position in $\mathbb{R}^d$, the alpha complex and Čech complex coincide.
6. Is there a metric on $\mathbb{R}^d$ such that Čech and Vietoris-Rips complexes coincide?

Useful tools:

- Given a simplicial complex X, define
$$f_i(X) = |\{\sigma \in X \mid \dim(\sigma) = i\}|, \quad -1 \leq i.$$
- Consider using the "extended" Euler characteristic, where we take into account also the unique (-1) - dimensional simplex $\emptyset$, defined by
$$\tilde{\chi}(X) = \sum_{i \geq -1} (-1)^i f_i(X).$$
Clearly, $\tilde{\chi}(X) = \chi(X) - 1$.
- We define the following polynomial
$$F(t, X) = \sum_{i \geq -1} (-1)^i f_i(X) t^{i+1}.$$
What is $F(1, X)$ equal to?