Essential Information Theory

PA154 Language Modeling (1.3)

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February 21, 2024

Source: Introduction to Natural Language Processing (600.465)
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The Notion of Entropy

- Entropy – "chaos", fuzziness, opposite of order,...
  - you know it
    - it is much easier to create"mess" than to tidy things up...
- Comes from physics:
  - Entropy does not go down unless energy is used
- Measure of **uncertainty**:
  - if low ...low uncertainty

Entropy

The higher the entropy, the higher uncertainty, but the higher "surprise" (information) we can get out of experiment.

The Formula

- Let $p_X(x)$ be a distribution of random variable X
- Basic outcomes (alphabet) Ω

Entropy

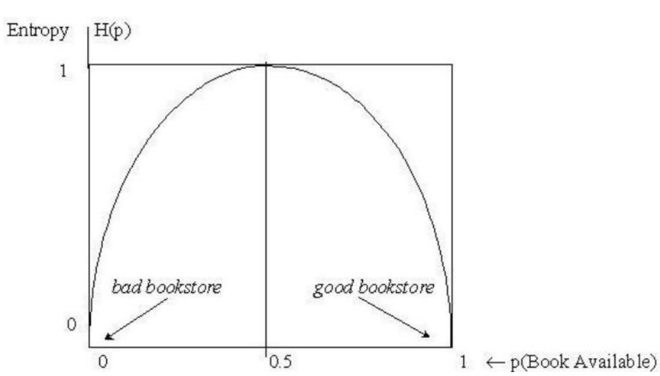
$H(X) = -\sum_{x\in\Omega} p(x)\log_2 p(x)$

- Unit: bits ($\log_{10}$: nats)
- Notation: $H(X) = H_p(X) = H(p) = H_X(p) = H(p_X)$

Using the Formula: Example

- Toss a fair coin: $\Omega = \{head, tail\}$
  - p(head) = .5, p(tail) = .5
  - $H(p) = -0.5\log_2(0.5) + (-0.5\log_2(0.5)) = 2 \times ((-0.5) \times (-1)) = 2 \times 0.5 = 1$
- Take fair, 32-sided die: $p(x) = \frac{1}{32}$ for every side x
  - $H(p) = -\sum_{i=1...32} p(x_i)\log_2 p(x_i) = -32(p(x_1)\log_2 p(x_1))$ (since for all i $p(x_i) = p(x_1) = \frac{1}{32}$ $= -32 \times (\frac{1}{32} \times (-5)) = 5$ (*now you see why it's called **bits**?*)
- Unfair coin:
  - p(head) = .2 ...**H(p) = .722**
  - p(head) = .01 ...**H(p) = .081**

Example: Book Availability



The Limits

- When $H(p) = 0$?
  - if a result of an experiment is ***known*** ahead of time:
  - necessarily:

$$\exists x \in \Omega; p(x) = 1 \& \forall y \in \Omega; y \neq x \Rightarrow p(y) = 0$$

- Upper bound?
  - none in general
  - for $|\Omega| = n : H(p) \leq \log_2 n$
    - nothing can be more uncertain than the uniform distribution

Entropy and Expectation

- Recall:
  - $E(X) = \sum_{x \in X(\Omega)} p_x(x) \times x$
- Then:

$E\left(\log_2\left(\frac{1}{p(x)}\right)\right) = \sum_{x \in X(\Omega)} p_x(x) \log_2\left(\frac{1}{p_x(x)}\right) =$
$-\sum_{x \in X(\Omega)} p_X(x) \log_2 p_x(x) = H(p_x) =_{notation} H(p)$

Perplexity: motivation

- Recall:
  - 2 equiprobable outcomes: H(p) = 1 bit
  - 32 equiprobable outcomes: H(p) = 5 bits
  - 4.3 billion equiprobable outcomes: H(p) ≅ 32 bits
- What if the outcomes are not equiprobable?
  - 32 outcomes, 2 equiprobable at 0.5, rest impossible:
    - H(p) = 1 bit
  - any measure for comparing the entropy (i.e. uncertainty/difficulty of prediction) (also) for random variables with *different number of outcomes*?

Perplexity

- Perplexity:
  - $G(p) = 2^{H(p)}$
- ...so we are back at 32 (for 32 eqp. outcomes), 2 for fair coins, etc.
- it is easier to imagine:
  - NLP example: vocabulary size of a vocabulary with uniform distribution, which is equally hard to predict
- the "wilder" (biased) distribution, the better:
  - lower entropy, lower perplexity

Joint Entropy and Conditional Entropy

- Two random variables: X (space Ω), Y (Ψ)
- Joint entropy:
  - no big deal: ((X,Y) considered a single event):

$$H(X,Y) = -\sum_{x \in \Omega}\sum_{y \in \Psi} p(x,y) \log_2 p(x,y)$$

- Conditional entropy:

$$H(Y|X) = -\sum_{x \in \Omega}\sum_{y \in \Psi} p(x,y) \log_2 p(y|x)$$

recall that $H(X) = E\left(\log_2 \frac{1}{p_x(x)}\right)$
(weighted "average", and weights are not conditional)

Conditional Entropy (Using the Calculus)

- other definition:

$H(Y|X) = \sum_{x \in \Omega} p(x) H(Y|X = x) =$
for $H(Y|X = x)$, we can use
the single-variable definition (x $\sim$ constant)
$= \sum_{x \in \Omega} p(x) \left(-\sum_{y \in \Psi} p(y|x) \log_2 p(y|x)\right) =$
$= -\sum_{x \in \Omega}\sum_{y \in \Psi} p(y|x)p(x) \log_2 p(y|x) =$
$= -\sum_{x \in \Omega}\sum_{y \in \Psi} p(x,y) \log_2 p(y|x)$

Properties of Entropy I

- Entropy is non-negative:
  - $H(X) \geq 0$
  - proof: (recall: $H(X) = -\sum_{x \in \Omega} p(x) \log_2 p(x)$)
    - $\log_2(p(x))$ is negative or zero for $x \leq 1$,
    - $p(x)$ is non-negative; their product $p(x)\log(p(x))$ is thus negative,
    - sum of negative numbers is negative,
    - and -f is positive for negative f
- Chain rule:
  - $H(X,Y) = H(Y|X) + H(X)$, as well as
  - $H(X,Y) = H(X|Y) + H(Y)$ (since $H(Y,X) = H(X,Y)$)

Properties of Entropy II

- Conditional Entropy is better (than unconditional):
  - $H(Y|X) \leq H(Y)$
- $H(X,Y) \leq H(X) + H(Y)$ (follows from the previous (in)equalities)
  - equality iff X,Y independent
  - (recall: X,Y independent iff p(X,Y)=p(X)p(Y))
- H(p) is concave (remember the book availability graph?)
  - concave function *f* over an interval (a,b):
    $\forall x, y \in (a,b), \forall \lambda \in [0,1]:$
    $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$
  - function *f* is convex if -*f* is concave
- for proofs and generalizations, see Cover/Thomas