LM Smoothing
(The EM Algorithm)

PA154 Language Modeling (2.3)

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Source: Introduction to Natural Language Processing (600.465)
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The Zero Problem

- "Raw" n-gram language model estimate:
  - necessarily, some zeros
    - !many: trigram model → 2.16×10^{14} parameters, data ~10^9 words
  - which are true 0?
    - optimal situation: even the least frequent trigram would be seen several times, in order to distinguish it's probability vs. other trigrams
    - optimal situation cannot happen, unfortunately *(open question: how many data would we need?)*
  - → we don't know
  - we must eliminate zeros
- Two kinds of zeros: p(w|h) = 0, or even p(h) = 0!

Why do we need Nonzero Probs?

- To avoid infinite Cross Entropy:
  - happens when an event is found in test data which has not been seen in training data

    H(p) = ∞: prevents comparing data with $\geq$ 0 "errors"
- To make the system more robust
  - low count estimates:
    - they typically happen for "detailed" but relatively rare appearances
  - high count estimates: reliable but less "detailed"

Eliminating the Zero Probabilites: Smoothing

- Get new p'(w) (same Ω): almost p(w) but no zeros
- Discount w for (some) $p(w) > 0$: new $p'(w) < p(w)$

$$\sum_{w \in discounted} (p(w) - p'(w)) = D$$

- Distribute D to all w; $p(w) = 0$: new $p'(w) > p(w)$
  - possibly also to other w with low p(w)
- For some w (possibly): $p'(w) = p(w)$
- Make sure $\sum_{w \in \Omega} p'(w) = 1$
- There are many ways of smoothing

Smoothing by Adding 1

Simplest but not really usable:

- Predicting words w from a vocabulary V, training data T:

$$p'(w|h) = \frac{c(h, w) + 1}{c(h) + |V|}$$

  - for non-conditional distributions: $p'(w) = \frac{c(w)+1}{|T|+|V|}$
- Problem if $|V| > c(h)$ (as is often the case; even $>> c(h)$!)

Example

Training data:	<s> what is it what is small? |T| = 8
V = {what, is, it, small, ?,<s> ,flying, birds, are, a, bird, .}, |V| = 12	
p(it) = .125, p(what) = .25, p(.)=0	p(what is it?) = $.25^2 \times .125^2 \cong .001$
	p(it is flying.) = $.125 \times .25 \times 0^2 = 0$
p'(it) = .1, p'(what) = .15, p'(.) = .05	p'(what is it?) = $.15^2 \times .1^2 \cong .0002$
	p'(it is flying.) = $.1 \times .15 \times .05^2 \cong .00004$

Adding *less than 1*

Equally simple:

- Predicting word w from a vocabulary V, training data T:

$$p'(w|h) = \frac{c(h, w) + \lambda}{c(h) + \lambda|V|}, \quad \lambda < 1$$

- for non-conditional distributions: $p'(w) = \frac{c(w)+\lambda}{|T|+\lambda|V|}$

Example

Training data:	<s> what is it what is small? |T| = 8
V = {what, is, it, small, ?,<s> ,flying, birds, are, a, bird, .}, |V| = 12	
p(it) = .125, p(what) = .25, p(.)=0	p(what is it?) = $.25^2 \times .125^2 \cong .001$
	p(it is flying.) = $.125 \times .25 \times 0^2 = 0$
Use λ = .1	
p'(it) $\cong$ .12, p'(what) $\cong$ .23, p'(.) $\cong$ .01	p'(what is it?) = $.23^2 \times .12^2 \cong .0007$
	p'(it is flying.) = $.12 \times .23 \times .01^2 \cong .000003$

Good-Turing

Suitable for estimation from large data

- similar idea: discount/boost the relative frequency estimate:

$$p_r(w) = \frac{(c(w)+1) \times N(c(w)+1)}{|T| \times N(c(w))}$$

where $N(c)$ is the count of words with count c (count-of-counts)
specifically, for $c(w) = 0$ (unseen words), $p_r(w) = \frac{N(1)}{|T| \times N(0)}$

- good for small counts (< 5–10, where $N(c)$ is high)
- normalization! (so that we have $\sum_w p'(w) = 1$)

Good-Turing: An Example

Remember: $p_r(w) = \frac{(c(w)+1) \times N(c(w)+1)}{|T| \times N(c(w))}$

Training data: <s> what is it what is small? |T| = 8
V = {what, is, it, small, ?,<s> ,flying, birds, are, a, bird, .}, |V| = 12
p(it) = .125, p(what) = .25, p(.)=0 p(what is it?) = $.25^2 \times .125^2 \cong .001$
p(it is flying.) = $.125 \times .25 \times 0^2 = 0$

- Raw estimation ($N(0) = 6, N(1) = 4, N(2) = 2, N(i) = 0$, for $i > 2$):
  p_r(it) = (1+1)×N(1+1)/(8×N(1)) = 2×2/(8×4) = .125
  p_r(what) = (2+1)×N(2+1)/(8×N(2)) = 3×0/(8×2) = 0:
  keep orig. p(what)
  p_r(.) = (0+1)×N(0+1)/(8×N(0)) = 1×4/(8×6) ≅ .083
- Normalize (divide by 1.5 = $\sum_{w \in |V|} p_r(w)$) and compute:
  p'(it) ≅ .08, p'(what) ≅ .17, p'(.) ≅ .06
  p'(what is it?) = $.17^2 \times .08^2 \cong .0002$
  p'(it is flying.) = $.08^2 \times .17 \times .06^2 \cong .00004$

Smoothing by Combination: Linear Interpolation

- Combine what?
  - distribution of various level of detail vs. reliability
- n-gram models:
  - use (n-1)gram, (n-2)gram, ..., uniform
    $\longrightarrow$ reliability
    $\longleftarrow$ detail
- Simplest possible combination:
  – sum of probabilities, normalize:
  - p(0|0) = .8, p(1|0) = .2, p(0|1) = 1, p(1|1) = 0,
    p(0) = .4, p(1) = .6
  - p'(0|0) = .6, p'(1|0) = .4, p'(0|1) = .7, p'(1|1) = .3

Typical n-gram LM Smoothing

- Weight in less detailed distributions using $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$:
  $p'_\lambda(w_i|w_{i-2}, w_{i-1}) = \lambda_3 p_3(w_i|w_{i-2}, w_{i-1}) +$
  $\lambda_2 p_2(w_i|w_{i-1}) + \lambda_1 p_1(w_i) + \lambda_0/|V|$
- Normalize:
  $\lambda_i > 0, \sum_{i=0}^{n} \lambda_i = 1$ is sufficient ($\lambda_0 = 1 - \sum_{i=1}^{n} \lambda_i$)(n = 3)
- Estimation using MLE:
  - <u>fix</u> the p_3, p_2, p_1 and |V| parameters as estimated from the training data
  - then find such $\{\lambda_i\}$ which minimizes the cross entropy (maximazes probablity of data): $-\frac{1}{|D|} \sum_{i=1}^{|D|} log_2(p'_\lambda(w_i|h_i))$

Held-out Data

- What data to use?
  – try training data T: but we will always get $\lambda_3 = 1$
  - why? let p_{iT} be an i-gram distribution estimated using r.f. from T)
  - minimizing $H_T(p'_\lambda)$ over a vector λ, p'_λ = $\lambda_3 p_{3T} + \lambda_2 p_{2T} + \lambda_1 p_{1T} + \lambda_0/|V|$
    – remember $H_T(p'_\lambda) = H(p_{3T}) + D(p_{3T}||p'_\lambda)$; p_{3T}fixed → $H(p_{3T})$ fixed, best)
    – which p'_λ minimizes $H_T(p'_\lambda)$? Obviously, a p'_λ for which $D(p_{3T}||p'_\lambda)$ = 0
    – ...and that's p_{3T} (because D(p||p) = 0, as we know)
    – ...and certainly $p'_\lambda = p_{3T}$if$\lambda_3 = 1$ (maybe in some other cases, too).
    – ($p'_\lambda = 1 \times p_{3T} + 0 \times p_{2T} + 1 \times p_{1T} + 0/|V|$)
  – thus: do <u>not use the training data for estimation of λ!</u>
  - must hold out part of the training data (***heldout*** data, <u>H</u>)
  - ...call remaining data the (true/raw) ***training*** data, <u>T</u>
  - the ***test*** data <u>S</u> (e.g., for comparison purposes): still different data!

The Formulas

Repeat: minimizing $\frac{-1}{|H|} \sum_{i=1}^{|H|} log_2(p'_\lambda(w_i|h_i))$ over λ

$$p'_\lambda(w_i|h_i) = p'_\lambda(w_i|w_{i-2}, w_{i-1}) = \\ = \lambda_3 p_3(w_i|w_{i-2}, w_{i-1}) + \lambda_2 p_2(w_i|w_{i-1}) + \lambda_1 p_1(w_i) + \lambda_0 \frac{1}{|V|}$$

"Expected counts of lambdas": j = 0..3

$$c(\lambda_j) = \sum_{i=1}^{|H|} \frac{\lambda_j p_j(w_i|h_i)}{p'_\lambda(w_i|h_i)}$$

"Next λ": j = 0..3

$$\lambda_{j,next} = \frac{c(\lambda_j)}{\sum_{k=0}^{3} c(\lambda_k)}$$

The (Smoothing) EM Algorithm

1. Start with some λ, such that $\lambda > 0$ for all j $\in$ 0..3
2. Compute "Expected Counts" for each λ_j.
3. Compute new set of λ_j, using "Next λ" formula.
4. Start over at step 2, unless a termination condition is met.

- Termination condition: convergence of λ.
  – Simply set an ε, and finish if $|\lambda_j - \lambda_{j,next}| < \varepsilon$ for each j (step 3).
- Guaranteed to converge: follows from Jensen's inequality, plus a technical proof.

Remark on Linear Interpolation Smoothing

- "Bucketed Smoothing":
  – use several vectors of λ instead of one, based on (the frequency of) history: λ(h)
  - e.g. for h = (micrograms,per) we will have
    λ(h) = (.999, .0009, .00009, .00001)
    (because "cubic" is the only word to follow...)

  – actually: not a separate set for each history, but rather a set for "similar" histories ("bucket"):
  λ(b(h)), where b: $V^2 \to N$ (in the case of trigrams)
  b classifies histories according to their reliability (~frequency)

Bucketed Smoothing: The Algorithm

- First, determine the bucketing function b (use heldout!):
  – decide in advance you want e.g. 1000 buckets
  – compute the total frequency of histories in 1 bucket (f_{max}(b))
  – gradually fill your buckets from the most frequent bigrams so that the sum of frequencies does not exceed f_{max}(b) (you might end up with slightly more than 1000 buckets)
- Divide your heldout data according to buckets
- Apply the previous algorithm to each bucket and its data

Simple Example

- Raw distribution (unigram only; smooth with uniform):
  p(a) = .25, p(b) = .5, p(α) = 1/64 for $\alpha \in \{c..r\}$, = 0 for the rest: s, t, u, v, w, x, y, z
- Heldout data: baby; use one set of λ
  (λ_1: unigram, λ_0: uniform)
- Start with $\lambda_0 = \lambda_1 = .5$:
  $$p'_\lambda(b) = .5 \times .5 + .5/26 = .27$$
  $$p'_\lambda(a) = .5 \times .25 + .5/26 = .14$$
  $$p'_\lambda(y) = .5 \times 0 + .5/26 = .02$$
  c(λ_1) = .5×.5/.27 + .5×.25/.14 + .5×.5/.27 + .5×0/.02 = 2.27
  c(λ_0) = .5×.04/.27 + .5×.04/.14 + .5×.04/.27 + .5×.04/.02 = 1.28
  Normalize $\lambda_{1,next}$ = .68, $\lambda_{0,next}$ = .32
  Repeat from step 2 (recompute p'_λ first for efficient computation, then c(λ_i), ...).
  Finish when new lambdas almost equal to the old ones (say, < 0.01 difference).

Some More Technical Hints

- Set V = {all words from training data}.
  - You may also consider V = T ∪ H, but it does not make the coding in any way simpler (in fact, harder).
  - But: you must *never* use the test data for your vocabulary
- Prepend two "words" in front of all data:
  - avoids beginning-of-data problems
  - call these index -1 and 0: then the formulas hold exactly
- When c_n(w,h) = 0:
  - Assing 0 probability to p_n(w|h) where c_{n-1}(h) > 0, but a uniform probablity (1/|V|) to those p_n(w|h) where c_{n-1}(h) = 0 (this must be done both when working on the heldout data during EM, as well as when computing cross-entropy on the test data!)

Back-off model

- Combines n-gram models
- using lower order in not enough information in higher order
-
$$P_{bo}(w_i|w_{i-n+1}\dots w_{i-1}) =$$
$$\begin{array}{ll} = d_{w_{i-n+1}\dots w_i}\dfrac{C(w_{i-n+1}\dots w_{i-1}w_i)}{C(w_{i-n+1}\dots w_{i-1})} & \text{if } C(w_{i-n+1}\dots w_i) > k \\ = \alpha_{w_{i-n+1}\dots w_{i-1}} P_{bo}(w_i|w_{i-n+2}\dots w_{i-1}) & \text{otherwise} \end{array}$$