2007 - Exercises I.

1. Consider the following general checksum scheme. Given is a fixed weighting vector $(w_1, \ldots, w_k)$. A number is encoded with a vector $(a_1, \ldots, a_k)$ so that

   $$a_1 w_1 + \ldots a_k w_k \equiv 0 \bmod n$$

   for some integer n greater than any possible a_i.

   Prove the following statements.

   (a) The checksum scheme can detect a single error occuring in position i if and only if w_i is relatively prime to n.

   (b) The checksum scheme can detect a transposition error occuring in positions i and j if and only if $|w_i - w_j|$ is relatively prime to n.

2. Let n, m, d be positive integers. Show that $A_2(nm, dm) \geq A_2(n, d)$.

3. (a) Find a binary $(10, 6, 6)$-code.

   (b) Find a binary $(5, 4, 3)$-code.

   (c) Let m be a positive integer. Show that

   $$A_2(5m, 3m) = \begin{cases} 6 & \text{if } m \text{ is even} \\ 4 & \text{otherwise.} \end{cases}$$

   Use the following inequality (Plotkin bound):

   $$A_2(n, d) \leq \begin{cases} 2\lfloor \frac{d}{2d-n} \rfloor & \text{if } d \text{ is even and } n < 2d \\ 2\lfloor \frac{d+1}{2d-n+1} \rfloor & \text{if } d \text{ is odd and } n < 2d+1. \end{cases}$$

4. Let C be a binary $(9, 6, 5)$-code which is transmitted over a binary symmetric channel with error probability $p = 0.01$ using the nearest neighbour decoding strategy. Find an upper bound of the word error probability for any codeword.

5. Design a Huffman coding for the following alphabet. Probabilities of occurence of each character are given.

   $$\left\{ (a, \frac{1}{8}), (b, \frac{1}{8}), (c, \frac{1}{8}), (d, \frac{1}{8}), (e, \frac{1}{8}), (f, \frac{1}{8}), (g, \frac{1}{8}), (h, \frac{1}{8}) \right\}$$

   What is the average length of encoding? (*i.e.* How long is the encoding of a text with n characters?)

6. Show that the code $C = \{0^n, 1^n\}$ is perfect if and only if n is odd.

7. Let us have a message $m = 01012414$ over a 5-ary alphabet.
Let us have an error correction code

$$C = \{0 \mapsto 01234, 1 \mapsto 12340, 2 \mapsto 23401, 3 \mapsto 34012, 4 \mapsto 40123\}$$

Let us have a channel X over 5-ary alphabet. For p being a probability of error and x a character from $\{0, 1, 2, 3, 4\}$:

$x \mapsto x$	with probability $1 - p$
$x \mapsto x + 1 \bmod 5$	with probability $p/4$
$x \mapsto x + 2 \bmod 5$	with probability $p/4$
$x \mapsto x + 3 \bmod 5$	with probability $p/4$
$x \mapsto x + 4 \bmod 5$	with probability $p/4$

We have sent m encoded by C through X and received

$$0120000110012301223022401444233333340023$$

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(a) Decode the message.

(b) Is code C always able to correct errors induced by channel X? If not, design a code that can do it.

(c) Can the code be more efficient if used on a modified channel X' which works over 6-ary alphabet?