2007 - Exercises III.

1. Determine whether the following code are cyclic:
   (a) $C_1 = \{x \mid x \in \mathbb{Z}_3^7 \text{ and } w(x) = 0 \pmod 3\}$
   (b) $C_2 = \{x \mid x \in \mathbb{Z}_3^7 \text{ and } x_1 + \ldots + x_7 = 0 \pmod 3\}$
   (c) $C_3 = \{x \mid x \in \mathbb{Z}_6^5 \ \text{ and } x = i \cdot 22222, i \in \mathbb{Z}_6\}$
2. Let C be the smallest binary cyclic code which contains the word 011011.
   (a) List the codewords of C.
   (b) Determine the polynomial $g(x)$ which generates C.
   (c) Use $g(x)$ to encode a message 11.
3. Find two different binary cyclic codes of length 8 which are equivalent or show that such codes do not exist.
4. (a) Factorize $x^6 - 1 \in \mathbb{F}_3[x]$ into irreducible polynomials.
   (b) Let n_k be the number of ternary cyclic codes of length 6 and dimension k. Determine n_k for $k \in \{0, 1, \ldots, 6\}$.
   (c) For each cyclic code of dimension 5, find the check polynomial and a parity check matrix and determine whether it contains the word 120210.
5. Let C_1 and C_2 be linear cyclic codes with generator polynomials $g_1(x)$ and $g_2(x)$. Show that the following code is linear and cyclic and find its generator polynomial.
$$C = \{c_1 + c_2 \mid c_1 \in C_1,\, c_2 \in C_2\}$$
6. Let C be a q-ary cyclic code of length n and let $f(x)$ be its generator polynomial. Show that all the codewords $c_0c_1 \ldots c_{n-1} \in C$ satisfy $c_0 + c_1 \cdots + c_{n-1} = 0$ in $\mathbb{F}_q$ if and only if the polynomial $x - 1$ is a factor of $f(x)$ in $\mathbb{F}_q[x]$.
7. Let C_1 and C_2 be cyclic codes of the same length over $\mathbb{Z}_q$. Is C a cyclic code?
   (a) $C = \{c_1c_2 \mid c_1 \in C_1, c_2 \in C_2\}$, where c_1c_2 denotes the concatenation of the codewords;
   (b) $C = C_1 \cup C_2$
   (c) $C = C_1 \cap C_2$
   (d) $C = C_1 \cup C_2^{\perp}$
   (e) $C = \Sigma^\star \setminus C_1$, where $\Sigma^\star$ is the set of all strings over $\mathbb{Z}_q$ of the same length as codewords of C_1;
   (f) $C = \{c_1 \oplus c_2 \mid c_1 \in C_1, c_2 \in C_2\}$, where $\oplus$ denotes characterwise addition modulo q.