$$y'' + y' - 2y = 0$$

char. pol.: $\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = -2$

ob. řeš.: $y = c_1 \cdot e^{\lambda_1 x} + c_2 \cdot e^{\lambda_2 x} =$

$$= c_1 e^{x} + c_2 e^{-2x}$$

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zk: $y = c_1 e^{x} + c_2 e^{-2x}$

$y' = c_1 e^{x} - 2c_2 e^{-2x}$

$y'' = c_1 e^{x} + 4c_2 e^{-2x}$

$y'' + y' - 2y =$

$= c_1 e^{x} + 4c_2 e^{-2x} +$

$+ c_1 e^{x} - 2c_2 e^{-2x} -$

$-2c_1 e^{x} - 2c_2 e^{-2x} = \underline{\underline{0}}$

$$y^{(4)} + 6y^{(3)} + 9y^{(2)} = 0$$

$$\lambda^4 + 6\lambda^3 + 9\lambda^2 = 0$$

$$\lambda^2 \cdot (\lambda^2 + 6\lambda + 9) = 0$$

$$\lambda^2 \cdot (\lambda + 3)^2 = 0 \iff \underbrace{\lambda_1 = 0}_{\text{DVOJNÁS.}},\ \underbrace{\lambda_2 = -3}_{\text{DVOJNÁS.}}$$

$$\Rightarrow y = c_1 \cdot e^{\lambda_1 x} + c_2 \cdot x \cdot e^{\lambda_1 x} + c_3 \cdot e^{\lambda_2 x} + c_4 \cdot x \cdot e^{\lambda_2 x} =$$

$$= c_1 + c_2 \cdot x + c_3 \cdot e^{-3x} + c_4 \cdot x \cdot e^{-3x}$$

NEHOM. ROVNICE

① VYŘEŠÍME PŘIDR. HOM. RCI $\Rightarrow y(x)$

② NAJDEME PART. ŘEŠ. NEHOM. RCE $\Rightarrow \eta(x)$

③ OB. ŘEŠ. NAŠÍ RCE $= y(x) + \eta(x)$

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a) PRAVÁ STR. $= \underbrace{Q_m(x)}_{\text{POLY st. } m} \cdot e^{\alpha x}$

$$\Rightarrow \eta(x) = x^{\rho} \cdot \underbrace{\overline{Q}_m(x)}_{\text{POLY st. } m \text{ (NEZNÁMÝ)}} \cdot e^{\alpha x}$$

ρ ... NÁS. α JAKOŽTO KOŘ. CHAR. POLY

$$y'' + 3y' + 2y = (20x+29)\cdot e^{3x}$$

$$\boxed{\begin{array}{l} Q_m(x) = 20x+29 \\ \alpha = 3 \end{array}}$$

1. HOM. RCE.: $y'' + 3y' + 2y = 0$

$$\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda_1 = -1,\ \lambda_2 = -2$$

$$y(x) = C_1\cdot e^{-x} + C_2\cdot e^{-2x}$$

2. $y_{(x)} = x^0\cdot(Ax+B)\cdot e^{3x} =$

$$\boxed{\alpha = 3 \Rightarrow \rho = 0}$$

$$= (Ax+B)\cdot e^{3x}$$

$$y'_{(x)} = A\cdot e^{3x} + (Ax+B)\cdot e^{3x}\cdot 3 = e^{3x}\cdot(A+3B+3Ax)$$

$$y''_{(x)} = e^{3x}\cdot 3\cdot(A+3B+3Ax) + e^{3x}\cdot 3x =$$

$$= e^{3x}\cdot(3A+9B+9Ax+3x)$$

$$\cancel{e^{3x}} \cdot (6A + 9Ax + 9B) + 3 \cdot \cancel{e^{3x}} \cdot (A + 3Ax + 3B) +$$

$$+ 2 \cdot \cancel{e^{3x}} \cdot (Ax + B) = (20x + 29) \cdot \cancel{e^{3x}}$$

x^0: $\quad 6A + 9B + 3A + 9B + 2B = 9A + 20B = 29$

x^1: $\quad 9A + 9A + 2A = 20A = 20 \Rightarrow \underline{\underline{A = 1}}$

$$9 \cdot 1 + 20B = 29$$

$$\underline{\underline{B = 1}}$$

$$\underline{y_{(x)} = (x+1) \cdot e^{3x}}$$

(3)

ob.ř. $= \underline{\underline{c_1 e^{-x} + c_2 e^{-2x} + (x+1) \cdot e^{3x}}}$

b) PR. STR. $= e^{\alpha x} \cdot (P(x) \cdot \cos \beta x + Q(x) \cdot \sin \beta x)$

$$\Rightarrow y(x) = x^{\rho} \cdot e^{\alpha x} \cdot (P_m(x) \cdot \cos \beta x + Q_m(x) \cdot \sin \beta x)$$

ρ ... NÁS. $\alpha + i\beta$ JAKO ŘEŠ. CHAR. POLY

NEZN. POLY st. m

$m = \max \{ st.(P), st.(Q) \}$

$$y'' - 3y' + 2y = 10\cos x$$

$\alpha = 0$, $\beta = 1$

$P(x) = 10$, $Q(x) = 0$

$\alpha + i\beta = \underline{\underline{i}}$

$\Downarrow$

$\rho = 0$

① Hom. RCE:

$$y'' - 3y' + 2y = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$D = 9 - 8 = 1$$

$$\lambda_{1,2} = \frac{3 \pm 1}{2} = \begin{cases} 2 = \lambda_1 \\ 1 = \lambda_2 \end{cases} \Rightarrow \underline{\underline{y(x) = c_1 e^{2x} + c_2 e^{x}}}$$

② PART. ŘEŠ.:

$$y_{(x)} = x^0 \cdot e^{0x} \cdot (A \cdot \cos x + B \cdot \sin x) =$$

$$= A \cos x + B \cdot \sin x$$

$$y'_{(x)} = -A \cdot \sin x + B \cdot \cos x$$

$$y''_{(x)} = -A \cdot \cos x - B \cdot \sin x$$

DOS. DO ŘCE $\Rightarrow$ $\underline{-A} \cos x \underline{-B} \sin x + \underline{3A} \sin x - \underline{3B} \cos x +$

$$+ \underline{2A} \cos x + \underline{2B} \sin x = 10 \cos x$$

$\sin x$: $-B + 3A + 2B = 3A + B = 0$ $\quad /\cdot(3)$

$\cos x$: $-A - 3B + 2A = A - 3B = 10$

$\oplus \Rightarrow 10A = 10$

$$\underline{\underline{A = 1}}$$

$\Rightarrow 3 \cdot 1 + B = 0$

$$\underline{\underline{B = -3}}$$

$y_{(x)} = \cos x - 3 \cdot \sin x$

$\Rightarrow$ ob. ř. $= c_1 e^{2x} + c_2 e^{x} + \cos x - 3 \cdot \sin x$

c) PR. STR. = SOUČET PR. STR. Z a) + b)

$\Rightarrow$ ↯ ZVLÁŠŤ & SEČTEM

$$y'' - 2y' = \underbrace{4x}_{a)} + \underbrace{2 \cdot \cos 2x}_{b)}$$

① <u>H.R.</u>: $y'' - 2y' = 0$

$$\lambda^2 - 2\lambda = 0$$

$$\lambda \cdot (\lambda - 2) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = 2$$

$$\Rightarrow \quad y(x) = c_1 + c_2 \cdot e^{2x}$$

a) $y'' - 2y' = 4x \quad \left(= Q_m(x) \cdot e^{\alpha x}\right)$

$\alpha = 0 \Rightarrow \rho = 1$

$Q_m(x) = 4x$

$$y_1(x) = x^1 \cdot (Ax + B) \cdot e^{0 \cdot x} =$$

$$= Ax^2 + Bx$$

$$y_1'(x) = 2Ax + B \quad , \quad y_1''(x) = 2A$$

$\rightarrow$ dos. : $2A - 4Ax - 2B = 4x$

x^0 : $2A - 2B = 0 \quad \Rightarrow B = -1$

x^1 : $-4A = 4 \quad \Rightarrow A = -1$

$$\Rightarrow \underline{\underline{y_1(x) = -x^2 - x}}$$

b) $y'' - 2y' = 2\cdot\cos 2x \quad (= e^{\alpha x}\cdot(P(x)\cdot\cos\beta x + Q(x)\cdot\sin\beta x))$

$$\boxed{\begin{array}{l}\alpha = 0,\ \beta = 2 \Rightarrow 0+2i \Rightarrow \rho = 0 \\ P(x) = 2,\quad Q(x) = 0\end{array}}$$

$y_2(x) = x^0\cdot e^{0x}\cdot(A\cdot\cos 2x + B\cdot\sin 2x) =$

$= A\cdot\cos 2x + B\sin 2x$

$y_2'(x) = -2A\sin 2x + 2B\cos 2x$

$\underline{y_2''(x) = -4A\cos 2x - 4B\sin 2x}$

$\underline{\underline{\text{DOS.}}} \Rightarrow -4A\cos 2x - 4B\sin 2x + 4A\sin 2x - 4B\cos 2x =$

$= 2\cdot\cos 2x$

$\left.\begin{array}{ll}\sin 2x: & -4B + 4A = 0 \\ \cos 2x: & -4A - 4B = 2\end{array}\right\}$ $\oplus$ $-8B = 2 \Rightarrow \underline{\underline{B = -\frac{1}{4}}}$

$\underline{\underline{A}} = B = \underline{\underline{-\frac{1}{4}}}$

$$y_2(x) = -\frac{1}{4}\cos 2x - \frac{1}{4}\sin 2x$$

CELKEM OB. ŘEŠ. =

$$\underbrace{c_1 + c_2 e^{2x}}\ \underbrace{-x - x^2}\ \underbrace{-\frac{1}{4}(\cos 2x + \sin 2x)}$$

METODA VARIACE KONSTANT

$$y'' + 2y' + y = \underbrace{e^{-x} \ln x}_{=: f(x)}$$

HOM. RCE: $y'' + 2y' + y = 0$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$(\lambda+1)^2 = 0 \iff \lambda = -1 \quad (\text{DVOJNÁS.})$$

OB. ŘEŠ. HOM. RCE $= c_1 \cdot e^{-x} + c_2 \cdot x \cdot e^{-x}$

$$\boxed{y_1 = e^{-x}, \quad y_2 = x \cdot e^{-x}}$$

$$y(x) = c_1(x)\cdot e^{-x} + c_2(x)\cdot x\cdot e^{-x}$$

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \cdot \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

$$(x\cdot e^{-x})' =$$

$$= 1\cdot e^{-x} +$$

$$+ x\cdot e^{-x}\cdot(-1) =$$

$$= e^{-x} - x\cdot e^{-x}$$

$$\begin{pmatrix} e^{-x} & x\cdot e^{-x} \\ -e^{-x} & e^{-x} - x\cdot e^{-x} \end{pmatrix} \cdot \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-x}\cdot \ln x \end{pmatrix}$$

$$\Rightarrow \quad c_1' \cdot \cancel{e^{-x}} + c_2' \cdot x \cdot \cancel{e^{-x}} = 0$$

$$-\cancel{e^{-x}} \cdot c_1' + \cancel{e^{-x}} \cdot (1-x) \cdot c_2' = \cancel{e^{-x}} \ln x$$

---

(1) $c_1' + c_2' x = 0$

(2) $-c_1' + (1-x) c_2' = \ln x$

$\left.\right\} \oplus \Rightarrow \quad c_2' \cdot (x + 1 - x) = \ln x$

$c_2' = \ln x$

$$c_2(x) = \int 1 \cdot \ln x \, dx = \left| \begin{array}{ll} u = \ln x & u' = \frac{1}{x} \\ v' = 1 & v = x \end{array} \right| =$$

$$= x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx = x \cdot \ln x - \int 1 \, dx =$$

$$= \underline{\underline{x \cdot \ln x - x + c_3}}$$

$(1) \Rightarrow \quad c_1' = -c_2' x = -x \cdot \ln x$

$$c_1(x) = -\int x \cdot \ln x \, dx = \left| \begin{matrix} u = \ln x & u' = \frac{1}{x} \\ v' = x & v = \frac{x^2}{2} \end{matrix} \right| =$$

$$= -\left[\frac{x^2}{2} \cdot \ln x - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx \right] =$$

$$= -\frac{x^2}{2} \ln x + \frac{1}{2} \int x \, dx = \underline{\underline{-\frac{x^2}{2} \ln x + \frac{1}{2} \frac{x^2}{2} + C_4}}$$

CELKEM:

$$y(x) = \left(-\frac{x^2}{2}\ln x + \frac{x^2}{4} + c_4\right)\cdot e^{-x} +$$

$$+ \left(x\cdot \ln x - x + c_3\right)\cdot x \cdot e^{-x} =$$

$$= \left(c_4 + c_3 x - \frac{3}{4}x^2 + \frac{x^2}{2}\ln x\right)\cdot e^{-x}$$

OPAKOVÁNÍ

$$\int \frac{3x^2+6x+2}{-x^4+3x^3-6x^2+12x-8}\,dx = ?$$

$$-x^4+3x^3-6x^2+12x-8 = (x-1)\cdot(-x^3+2x^2-4x+8) =$$

	-1	3	-6	12	-8
1	-1	2	-4	8	0 ✓

$$= (x-1)\cdot\left[x^2\cdot(2-x) + 4\cdot(2-x)\right] =$$

$$= (x-1)\cdot(2-x)\cdot(x^2+4)$$

$$\frac{3x^2+6x+2}{-x^4+3x^3-6x^2+12x-8} = \frac{A}{x-1} + \frac{B}{2-x} + \frac{Cx+D}{x^2+4} \quad \Big/ \cdot (\sim)$$

$$3x^2+6x+2 = A\cdot(2-x)\cdot(x^2+4) + B\cdot(x-1)\cdot(x^2+4) +$$

$$+ (Cx+D)\cdot(x-1)\cdot(2-x)$$

$$x=1 \Rightarrow \quad 11 = A\cdot(1)\cdot(5) + \cancel{B\cdot 0 + (C+D)\cdot 0}$$

$$A = \frac{11}{5}$$

$$x=2 \Rightarrow \quad 12+12+2 = 26 = \cancel{A\cdot 0} + B\cdot(1)\cdot(8) + \cancel{(2C+D)\cdot 0}$$

$$B = \frac{26}{8} = \frac{13}{4}$$

$$x=2i \Rightarrow \quad 3\cdot(2i)^2+6\cdot 2i+2=-12+12i+2=$$

$$=\underbrace{12i-10}= \cancel{A\cdot 0}+\cancel{B\cdot 0}+$$

$$+(C\cdot 2i+D)\cdot(2i-1)\cdot(2-2i)=$$

$$=(2iC+D)\cdot(4i+4-2+2i)=(2iC+D)\cdot(6i+2)=$$

$$=\underbrace{-12C+4iC+6iD+2D}$$

$$\left.\begin{array}{ll} Re: & -10=-12C+2D \Rightarrow D-6C=-5 \\ Im: & 12=4C+6D \Rightarrow 3D+2C=6 \end{array}\right\} \begin{array}{l} C=\frac{21}{20} \\ D=\frac{13}{10} \end{array}$$

$$(*) = \int \frac{\frac{11}{5}}{x-1}\,dx + \int \frac{\frac{13}{4}}{2-x}\,dx +$$

$$+ \underbrace{\int \frac{\frac{21}{10}x + \frac{13}{10}}{x^2+4}\,dx}_{} = (**)$$

$$\left| \begin{aligned} t &= \frac{x}{2} \\ dt &= \frac{dx}{2} \\ dx &= 2\,dt \end{aligned} \right|$$

$$\frac{21}{10}\cdot\frac{1}{2}\int\frac{2x}{x^2+4} + \frac{13}{10}\cdot\frac{1}{4}\int\frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx =$$

$$= \frac{21}{20}\cdot \ln(x^2+4) + \frac{13}{20}\cdot\int\frac{1}{t^2+1}\,dt =$$

$$= \frac{21}{20} \cdot \ln(x^2+4) + \frac{13}{20} \cdot \operatorname{arctg}\frac{x}{2} + C_1$$

$$(**) = \frac{11}{5} \cdot \ln|x-1| + \frac{13}{4} \cdot \ln|2-x| \cdot (-1) +$$

$$+ \frac{21}{20} \cdot \ln(x^2+4) + \frac{13}{20} \operatorname{arctg}\frac{x}{2} + C$$

$$\left[\ln x + \left(\cos e^{x^2}\right)^{\pi}\right]' =$$

$$= \frac{1}{x} + \pi \cdot \left(\cos e^{x^2}\right)^{\pi-1} \cdot \left(-\sin e^{x^2}\right) \cdot e^{x^2} \cdot 2x$$

$$\left[x^{\cos x}\right]' = \left[e^{\ln x^{\cos x}}\right]' =$$

$$= \left(e^{\cos x \cdot \ln x}\right)' = \underbrace{e^{\cos x \cdot \ln x}}_{x^{\cos x}} \cdot \left(\cos x \cdot \ln x\right)' =$$

$$= x^{\cos x} \cdot \left(-\sin x \cdot \ln x + \cos x \cdot \frac{1}{x}\right)$$