a)

$$f(x,y) = x^2y + y^2x - xy$$

$$\left.\begin{aligned} f_x &= 2xy + y^2 - y = 0 \\ f_y &= x^2 + 2xy - x = 0 \end{aligned}\right\} \quad (0,0),\ (0,1),\ (1,0),\ (1/3, 1/3)$$

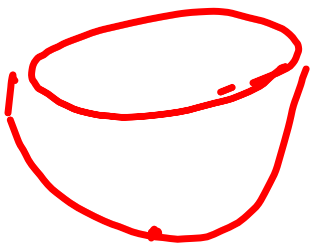
$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = H$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad 0,\ -1$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} \quad \boxed{\begin{matrix} 2/3 \\ 3/9 \end{matrix}}$$



2)

$$f(x,y) = \ln(x^2+y^2+1) \quad \sim \quad [1,1]$$

$$T_2 f(1,1) = f(1,1) + f_x(1,1)(x-1)$$
$$+ f_y(1,1)(y-1)$$
$$+ \frac{1}{2}(x-1,\, y-1)\cdot\begin{pmatrix} \frac{2}{9} & -4/9 \\ -4/9 & 2/9 \end{pmatrix}\cdot\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$$

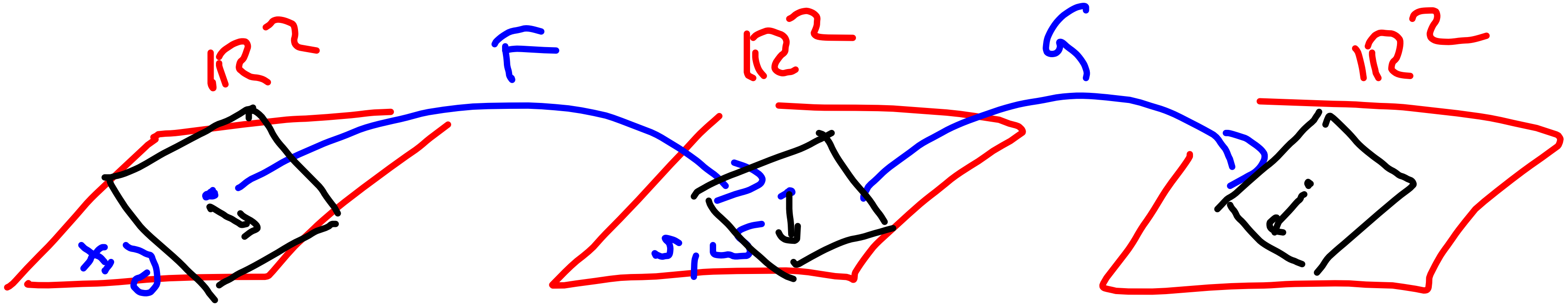
$$= \frac{1}{9}\left(x^2+y^2+8x+8y-4xy-15\right)$$
$$+ \ln(3)$$

$f_x = ?$ $\quad f_y = ?$

	a	f_x	f_y
$(\pm\frac{1}{\sqrt{2}}, 0)$	1	$\boxed{(-2x+1)e^{-x^2-y^2}}$	$-2xy\,e^{-x^2-y^2}$
$(0, y)$ $(\pm 1, 0)$	2	$(2x-2x^3)e^{-x^2-y^2}$	$-2x^2y\,e^{-x^2-y^2}$
$(0, y)$ $(\pm\sqrt{3/2}, 0)$	3	$(3x^2-2x^4)e^{-x^2-y^2}$	$-2x^3y\,e^{-x^2-y^2}$

Derivace složeného zobrazení:

$F: \mathbb{R}^2 \to \mathbb{R}^2 \qquad G: \mathbb{R}^2 \to \mathbb{R}^2$



$F = (f(x,y),\ g(x,y)) \qquad G = (p(r,s),\ q(r,s))$

$f = r, \quad g = s$

$G \circ F(x,y) = \big(p(f(x,y), g(x,y)),\ q(f(x,y), g(x,y))\big)$

$$D^1G = \begin{pmatrix} \frac{\partial p}{\partial r} & \frac{\partial p}{\partial s} \\ \frac{\partial q}{\partial r} & \frac{\partial q}{\partial s} \end{pmatrix}$$

$$D^1G(h_1, h_2) = \begin{pmatrix} \cdot & \\ & \cdot \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$D^1(G\circ F) = D^1 G \circ D^1 F$$

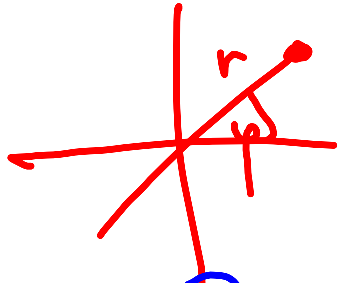
pro $F, G : \mathbb{R} \to \mathbb{R}$

$$\frac{d(F\circ G)}{dt} = F'(G(t_0))\cdot G'(t_0)$$

Příklad:

$(x,y) \mapsto (r,\varphi)$ (polární souřadnice).

$r = \sqrt{x^2+y^2}$, $\varphi = \operatorname{arctg} \frac{y}{x}$



$$g(r,\varphi,t) = \sin(r-t)$$

parametr

spočítejte

$$\boxed{\frac{\partial g}{\partial x}(x,y,t)} = \frac{\partial g}{\partial r}(r,\varphi)\frac{\partial r}{\partial x}(x,y) + \frac{\partial g}{\partial \varphi}(r,\varphi)\cdot\frac{\partial \varphi}{\partial x}(x,y)$$

$$= \cos\left(\sqrt{x^2+y^2}-t\right)\cdot\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} + 0$$

$$\boxed{\frac{\partial g}{\partial y}(x,y,t)} = \frac{\partial g}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial g}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial y}$$

zpříčte

$$= \cos\left(\sqrt{x^2+y^2} - t\right)\frac{y}{\sqrt{x^2+y^2}}$$

$g : \mathbb{R}^2 \to \mathbb{R}$, $F : \mathbb{R}^2 \to \mathbb{R}^2$

funkce

zobraz. → poloměr

$F: \mathbb{R}^2 \to \mathbb{R}^2$

$\mathbb{R}^n \to \mathbb{R}^n$

F je "bijekce" na okolí a diferencovatelná

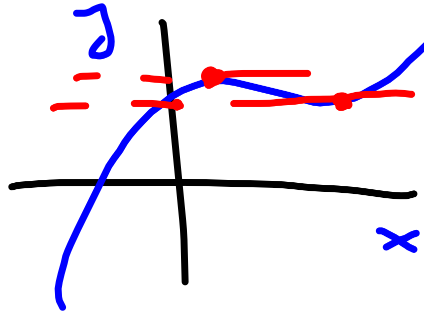
$\Leftrightarrow$ D^1F je <u>invertibilní</u>

$$F^{-1} \circ F = id$$

$$(D^1F^{-1}) \cdot (D^1F) = E$$

$$\Rightarrow (D^1F^{-1}) = (D^1F)^{-1}$$

neznáme   známe



dim 1

$y = f(x)$

$x \overset{?}{=} f^{-1}(y)$

ano, když $f'(x) \neq 0$ (na okolí)

$$(f^{-1})' = \frac{1}{f'}$$

$$r = \sqrt{x^2 + y^2}$$

$$\varphi = \operatorname{arctg} \frac{y}{x}$$

F^{-1}

$$x = r \cos\varphi$$

$$y = r \cdot \sin\varphi$$

$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad (r, \varphi) \to (x, y)$

$$(D^1 F)^{-1}$$

$$D^1 F = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}^{-1}$$

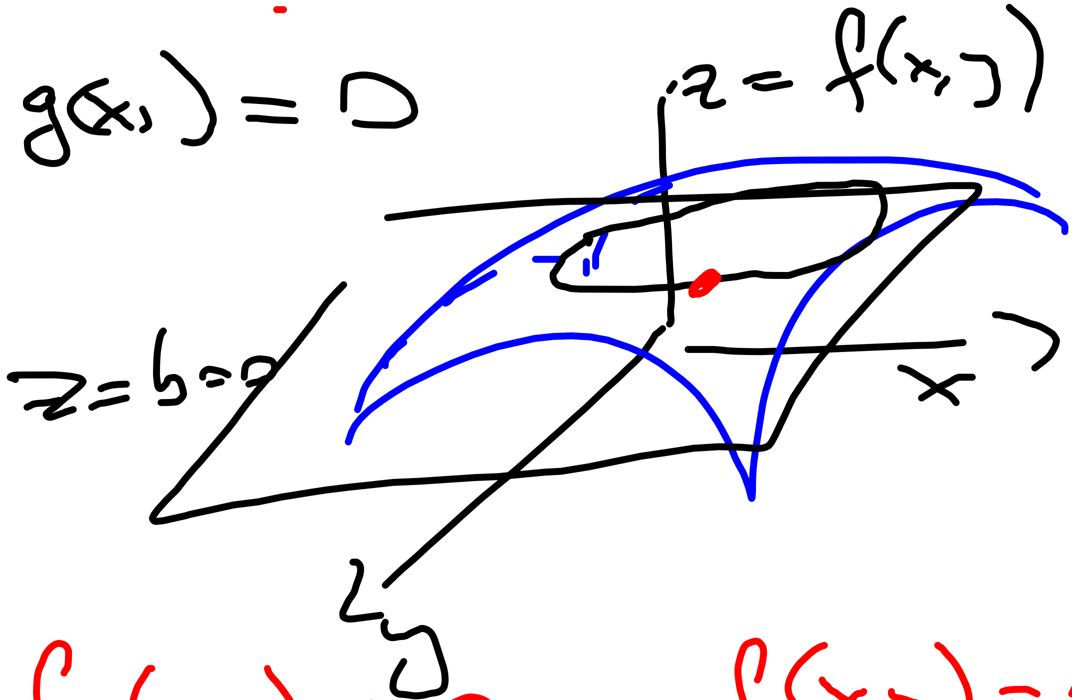
$$|D^1 F| = r(\cos^2\varphi + \sin^2\varphi) = r \neq 0$$

Spočítat $(D^1 F^{-1})$

$f : \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = 0 \overset{?}{\Longrightarrow} y = g(x)$ t.ž.,

aby $f(x, g(x)) = 0$



Pokud $f_y(x,y) \neq 0$ a $f(x,y) = 0$

$\Rightarrow$ existuje ve okolí x funkce $g(x)$ s

předem danými vlastnostmi.

nemá'c psal $\qquad f(x, y(x)) = 0$

$$0 = \frac{d}{dx} f(x, y(x)) = \frac{\partial f}{\partial x}(x, y(x)) + \frac{\partial f}{\partial y}(x, y(x))\, y'(x)$$

$$y'(x) = -\frac{f_x(x, y(x))}{f_y(x, y(x))} \neq 0$$

$$f_x = e^x \sin y + e^y \cos x$$

$$f_y = \underbrace{e^x}_{>0} \underbrace{\cos y}_{\geq 0} + \underbrace{e^y}_{>0} \underbrace{\sin x}_{\geq 0} > 0$$

$\Rightarrow$ $f(x,-)$ rostoucí a spojité na $\langle 0, \pi/2 \rangle$

$$f(x,0) = \sin x < 1$$

$$f(x,\pi/2) = \underbrace{e^x + e^{\pi/2} \sin x}_{\text{rostoucí}} > 1$$

$\Rightarrow$ $\forall x \in \langle 0, \pi/2 \rangle$ ex. právě jedno $y(x)$ tak, že $f(x,y) = 1$.

$$f'(x, y(x)) = -\frac{f_x}{f_y} = -\frac{e^x \sin y + e^y \cos x}{e^x \cos y + e^y \sin x}$$

$F(x,y,z) = \underline{\sin(xy)} + \underline{\sin(yz)} + \underline{\sin(xz)} - \underline{1}$

$z = z(x,y)$ tak aby $F(x,y,z(x,y)) = 0$

dokázání $F_z(x,y,z) \neq 0$ ✓, $\underline{F(x,y,z) = 0}$ ✓

po $(x,y,z) = (\sqrt{\pi/2}, \sqrt{\pi/2}, 0)$.

$$\frac{\partial F}{\partial z} = \cos(yz)\cdot y + \cos(xz)\cdot x \Big|_{(\underset{x}{\sqrt{\pi/2}},\ \underset{y}{\sqrt{\pi/2}},\ \underset{z}{0})} = \sqrt{\pi/2} + \sqrt{\pi/2} = 2\sqrt{\pi/2} \neq 0$$

$$F(\sqrt{\pi/2}, \sqrt{\pi/2}, 0) = 1 + 0 + 0 - 1 = 0$$

$$\frac{\partial F}{\partial x} = \cos(xy)\cdot y + \cos(zx)\cdot z$$

$$\frac{\partial F}{\partial y} = \cos(xy)\cdot x + \cos(yz)\cdot z$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = \frac{\cos(xy)\cdot y + \cos(zx)\cdot z}{\cdot/\cdot}$$

$$\Big|_{\left(\sqrt{\pi/2},\, \sqrt{\pi/2},\, 0\right)} = \frac{0 + 0}{2\sqrt{\pi/2}}$$

$$\frac{\partial z}{\partial y} = -\frac{0 + 0}{2\sqrt{\pi/2}} = 0$$

$F: \mathbb{R}^2 \to \mathbb{R}$

$\boxed{F = 0}$

$F: y - x^2 - 2x - 1 = 0$

$f(x,y): \mathbb{R}^2 \to \mathbb{R}$

jaké jsou extrémy f

na $\boxed{F=0}$

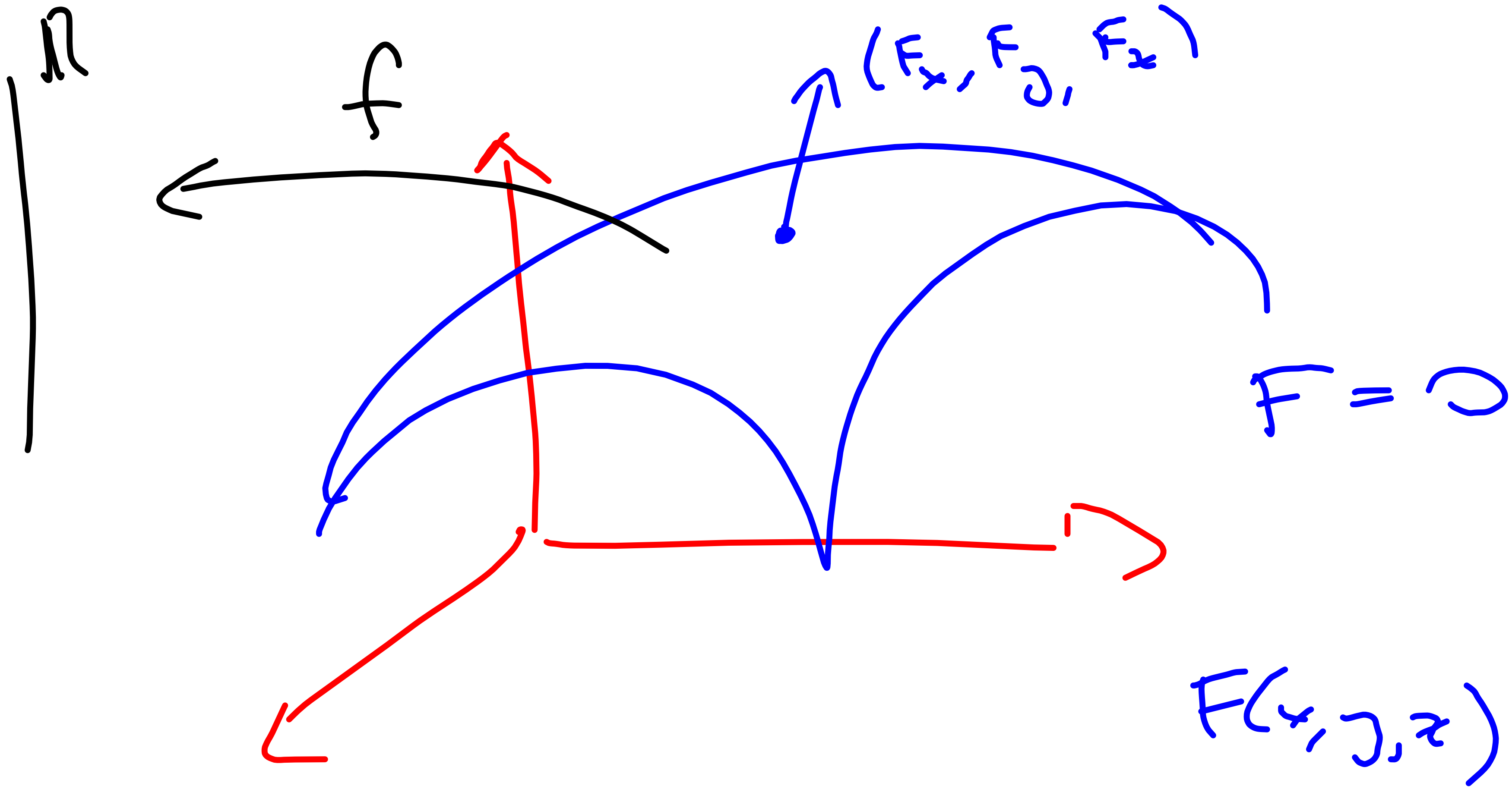
gradient F je

$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right)$

a to je směr, ve

kterém F nejrychleji

roste



f má „stacionární bod na $F = 0$"

$$\Longleftrightarrow \quad (f_x, f_y, f_z) = \lambda (F_x, F_y, F_z)$$

$$F(x,y) = y - x^3 - 2x - 1 \qquad F_x = -3x^2 - 2$$

$$F_y = 1$$

$$f(x,y) = x - 2y \qquad f_x = 1$$

$$f_y = -2$$

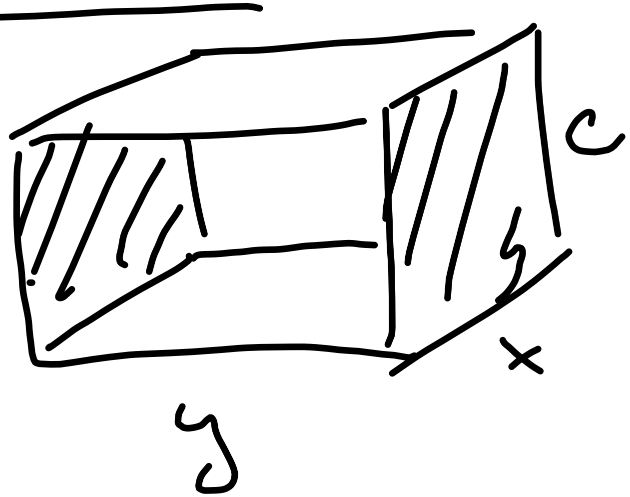
$$-2 = \lambda \cdot 1 \Rightarrow \lambda = -2$$

$$1 = -2(-3x^2 - 2) = 6x^2 + 4$$

není řešení $\Rightarrow$ není stac. bod

pro $x \in \langle 0, 5 \rangle$ spočteme
krajní 5, 0 5.

bedna :



$$xy = 10$$

$$F(x,y) = xy - 10$$

$$f(x,y) = 2\underbrace{xy}_{10} + c(2x+y)$$

$$F_x = y \qquad f_x = 2(y+c)$$

$$F_y = x \qquad f_y = 2x + c$$

$$\rightsquigarrow \boxed{\frac{y}{x} = 2}$$