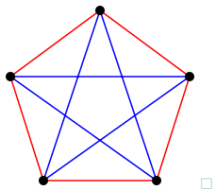


12 Introduction to Ramsey Theory

Informally saying, Ramsey theory is a theory of numerous results in different areas of discrete mathematics, all basically showing that “a complete disorder is mathematically impossible”. In other words, Ramsey theory is looking for order inside disorder.

We very briefly introduce some fundamental directions of research of this interesting branch of mathematics.



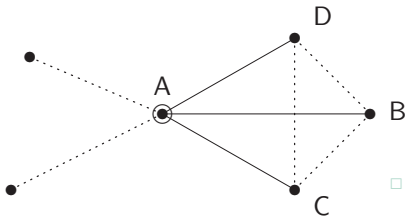
Brief outline of this lecture

- Exmpl. problems: the party problem, Schur's theorem, convex points
- Ramsey theorem: its infinite version, and some special finite bounds
- Hales-Jewett theorem – another kind of problems

12.1 Some exemplary problems

Example 12.1. Consider six people meeting at a party. Then three of them know each other, or three of them are mutually strangers. \square

Up to symmetry, person A knows at least three others (out of five) – say B,C,D.



Then either some two of B,C,D know each other (and we have three!), or B,C,D are mutually strangers. \square

Theorem 12.2. (the Party theorem) For every integer p there exists N such that: In any group of N people, some p of them know each other or in some group of p they are mutually strangers.

Remark. Imagine that the pairs of people of the group are coloured blue if they know each other, or red if they are strangers (do not know each other). Then Theorem 12.2 claims that there exists a *monochromatic k -tuple*.

Schur's theorem

Theorem 12.3. For any integer $c > 0$ there exists N such that the following holds: If the set of integers from 1 to N is coloured arbitrarily with c colours, then the equation $x + y = z$ has a **monochromatic solution** in this set. \square

Example 12.4. If the set $\{1, 2, 3, 4, 5\}$ is 2-coloured, then the equation $x + y = z$ has a monochromatic solution there (Theorem 12.3 for $c = 2$ and $N = 5$).

Up to symmetry, number 1 is red and then 2 is blue, hence 4 is red. . .

1, 2, 3, 4, 5 \square

What is the colour of 3? It must be blue. However, what is then the colour of 5?

1, 2, 3, 4, 5?

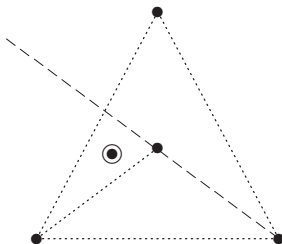
\square

Convex point configurations

Theorem 12.5. *For any integer $q \geq 3$ there exists N such that the following holds: If N points in the plane are in an arbitrary general position (i.e. no three of them are colinear), then there exist q of them which form the vertices of a convex q -gon. \square*

Example 12.6. *Among any 5 planar points in general position, some 4 form a convex quadrangle. \square*

This is an easy analysis, starting from the convex hull of all (a triangle)...



\square

12.2 Theorem(s) of Ramsey

Theorem 12.7. (Ramsey theorem – infinite) For any integers c and k the following holds: If each of the k -tuples of elements of an infinite sequence S is coloured with one of c colours, then there exists an infinite subsequence $S_0 \subseteq S$ such that all the k -tuples of elements of S_0 have the same colour (*monochromatic S_0*).

Remark: For $k = 1$, this is the pigeon-hole principle... \square

Proof: We use induction on k . The case of $k = 1$ is clear by the remark.

We now consider $(k + 1)$ -tuples. Set $S_1 = S$, and x_1 be the first element of S .

- Assign every k -tuple T of $S_1 \setminus \{x_1\}$ the colour of $T \cup \{x_1\}$. By the inductive assumption, there exists an infinite **monochromatic subsequence** $S'_1 \subseteq S_1 \setminus \{x_1\}$. \square
- Repeat the previous with $S_2 = S'_1$, then with S_3, S_4, \dots (“till infinity”) \square

Finally, consider the above selected sequence $S' = (x_1, x_2, x_3, \dots)$. Every x_i has its “own colour” (of the monochromatic subsequence S'_i), and one colour **repeats infinitely** many times. This is our S_0 . \square

Theorem 12.8. (Ramsey theorem)

For any integers c, k and n , there exists $N = R(k; c)$ such that the following holds: If each of the k -tuples of elements of an N -element set M is coloured with one of c colours, then there exists a subset $M_0 \subseteq M$ of n elements such that all the k -tuples of elements of M_0 have the same colour (*monochromatic M_0*).

This theorem follows from Theorem 12.7 via a usual compactness argument. However, this approach gives no usable bounds on $R(k; c)$. \square

Lemma 12.9. (Theorem 12.2) For every integers m, ℓ the following is true: In any collection of $N(m, \ell) = \binom{m+\ell-2}{m-1}$ people, some m of them know each other or in some group of ℓ they are mutually strangers. \square

Proof by induction on $m + \ell$ (a sketch):

Notice that (cf. Pascal's triangle)

$$N(m, \ell) = \binom{m + \ell - 2}{m - 1} = \binom{m + \ell - 3}{m - 2} + \binom{m + \ell - 3}{m - 1} = N(m - 1, \ell) + N(m, \ell - 1). \square$$

Pick any person A . Then either at least $N(m - 1, \ell)$ persons are known to A , or at least $N(m, \ell - 1)$ persons are strangers to A . In either case, by induction, we get a solution of the respective subproblem, and possibly add A to it. \square

Edge-coloured graph view

Ramsey-type results for coloured pairs ($k = 2$) are often formulated for edge-coloured (not proper colourings, though!) graphs. Such as the Party problem (Theorem 12.2) asks for a monochromatic copy of K_p . Or, the following is well-known:

Lemma 12.10. *For any $c > 1$ the following holds: If the edges of a complete graph K_N where $N = N(c) = \lceil c! \cdot e \rceil$ are coloured with c colours, then there exists a monochromatic triangle. (Here $e = 2.718281\dots$) \square*

Proof: For $c = 2$, this is true by Example 12.1. We continue the proof by induction on $c \geq 2$; taking $N_1 = (c + 1) \cdot (N - 1) + 2$ where $N = N(c)$.

Pick any vertex a of K_{N_1} (whose edges are coloured with $c + 1$ colours). Then a has $N_1 - 1 = (c + 1) \cdot (N - 1) + 1$ incident edges, and at least N of them, by the pigeon-hole principle, have the same colour – say 1. The subgraph induced on the other ends of these edges is a copy K' of K_N . So, either some edge of this K' has colour 1 and we get a triangle in this colour with a , or K' is edge- c -coloured, and the statement follows by induction. \square

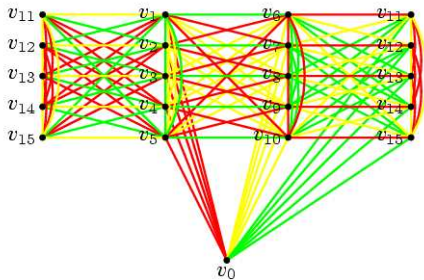
Lastly, it remains to verify that $N(c + 1) = \lceil (c + 1)! \cdot e \rceil = \lfloor c! \cdot e \rfloor \cdot (c + 1) + 2$, which follows from well-known $e = \sum_{i=0}^{\infty} 1/i!$. \square

Graph Ramsey numbers

Definition: The *Ramsey number* $R(\underbrace{K_p, K_q, K_r, \dots}_c)$ is the minimum integer R such that the following is true: Whenever the edges of the complete graph K_R are coloured arbitrarily with c colours, then there exists a monochromatic copy of K_p in colour 1, or of K_q in colour 2, or of K_r in colour 3, or \dots

This Ramsey number is always defined by Theorem 12.8, and the definition can be extended to arbitrary subgraphs instead of the cliques. \square

For instance, $N(c) = R(\underbrace{K_3, K_3, \dots}_c)$ from Lemma 12.10 are tight for $c = 2, 3$ ($N = 6, 17$).

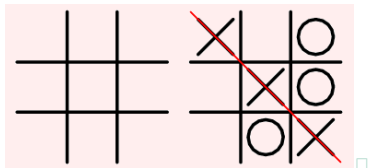


Corollary 12.11. (Theorem 12.3) For any integer $c > 0$ the following holds: If the set of integers from 1 to $N(c) = \lceil c! \cdot e \rceil$ is coloured arbitrarily with c colours, then the equation $x + y = z$ has a *monochromatic solution* in this set. \square

Proof: We colour the edges of the complete graph K_N on vertices $\{1, \dots, N\}$ as follows: $\{u, v\}$ gets the colour of the number $|u - v|$. Then a monochromatic triangle exists by Lemma 12.10 – say on the vertices $a < b < c$ – and hence the numbers $x = b - a$, $y = c - b$, and $z = c - a$ have the same colour and $x + y = z$. \square

12.3 Hales-Jewett theorem

Informal explanation: in high-dimensional tic-tac-toe game, there cannot be a draw.



Definition: Let n and D be integers. A *formal hypercube* of dimension D and side n is the set of formal words $\Sigma^D = \{1, 2, \dots, n\}^D$.

A *formal (combinatorial) line* in Σ^D is a word $\ell \in (\Sigma \cup \{x\})^D \setminus \Sigma^D$ where x is a (symbolic) variable. The n points of this line are the words $\ell(x=1), \dots, \ell(x=n)$. \square

Remark: Every formal line is a line in the “usual sense”, but not the converse. Hence the coming statement is actually much stronger than just saying that a high-dimensional tic-tac-toe game has no draw.

Theorem 12.12. (Hales-Jewett) *For every integers $n, c > 1$ there exists $D = D(n, c)$ such that the following holds: If the points of a formal D -dimensional hypercube of side n are coloured with c colours, then there always exists a monochromatic combinatorial line.*

Van der Waerden's theorem

Theorem 12.13. (van der Waerden) *For every integers $n, c > 1$ there exists $W = W(n, c)$ such that the following holds: If the integers $1, 2, \dots, W$ are coloured with c colours, then there exists a monochromatic arithmetic progression among them. \square*

Proof : We choose $W = n^{D(n,c)}$ where D is from Theorem 12.12. The integers $1, 2, \dots, W$, written down as n -adic numbers, then form a hypercube of side n and dimension $D(n, c)$. A monochromatic line exists in this hypercube by Theorem 12.12, and the points of this line clearly give an arithmetic progression of length n .