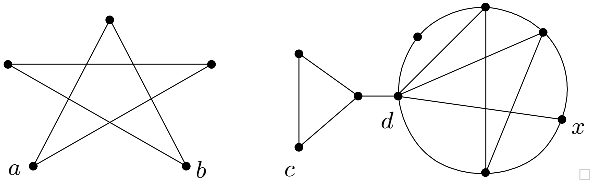
3 Distances in Graphs

While the previous lecture studied mainly bare graph connectivity, now we are going to investigate how "long" a connection could be.

This naturally leads to the concept of graph distance, which has two variants: a simple one only measures by the number of edges, while the weighted one has a "length" for each edge.



Brief outline of this lecture

- Distance in a graph, basic properties.
- Graph metrics, and a dynamic computation of it (Floyd–Warshall).
- Dijkstra's algorithm for the shortest weighted distance in a graph.

3.1 Graph distance

Recall that a walk of length n in a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$ such that each e_i has edns v_{i-1}, v_i.

Definition 3.1. **Distance** $d_G(u,v)$ between two vertices u, v of a graph G is defined as the length of the shortest walk between u and v in G.
If there is now walk between u, v, then we declare $d_G(u,v) = \infty$. □

Informally and naturally, the distance between u, v equals *the least possible number of edges* traversed from u to v. Specially $d_G(u,u) = 0$.

Recall, moreover, that the shortest walk is always a path – Theorem 2.2.

Fact: The distance in an undirected graph is symmetric, i.e. $d_G(u,v) = d_G(v,u)$. □

Lemma 3.2. *The graph distance satisfies the triangle inequality:*

$$\forall u, v, w \in V(G) \ : \ \ d_G(u,v) + d_G(v,w) \geq d_G(u,w)\,.$$

Proof. Easily; starting with a walk of length $d_G(u,v)$ from u to v, and appending a walk of length $d_G(v,w)$ from v to w, results in a walk of length $d_G(u,v) + d_G(v,w)$ from u to w. This is an upper bound on the real distance from u to w. □

How to find the distance

Theorem 3.3. *Let u, v, w be vertices of a connected graph G such that $d_G(u, v) < d_G(u, w)$. Then the breadth-first search algorithm on G, starting from u, finds the vertex v before w.* □

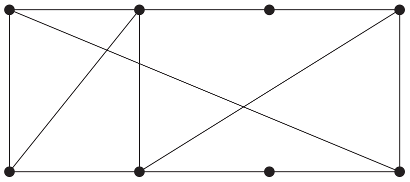
Proof. We apply induction on the distance $d_G(u, v)$: If $d_G(u, v) = 0$, i.e. $u = v$, then it is trivial that v is found first. So let $d_G(u, v) = d > 0$ and v' be a neighbour of v closer to u, which means $d_G(u, v') = d - 1$. Analogously choose w' a neighbour of w closer to u. Then

$$d_G(u, w') \geq d_G(u, w) - 1 > d_G(u, v) - 1 = d_G(u, v'),$$

and so v' has been found before w' by the inductive assumption. Hence v' has been stored into U before w', and (cf. FIFO) neighbours of v' are found before neighbours of w'. □ □

Corollary 3.4. *Breadth-first search algorithm on G correctly determines graph distances from the starting vertex.*

Other related terms



Definition. Let G be a graph. We define, with respect to G, the following notions:

- The *excentricity* of a vertex exc(v) is the largest distance from v to another vertex; exc$(v) = \max_{x \in V(G)} d_G(v, x)$. □
- The *diameter* diam(G) of G is the largest excentricity over its vertices, and the *radius* rad(G) of G is the smallest excentricity over its vertices. □
- The *center* of G is the subset $U \subseteq V(G)$ of vertices such that their excentricity equals rad(G).

3.2 Computing all-pairs distances

Definition: The *metrics* of a graph is the collection of distances between all pairs of its vertices. In other words, the metrics is a matrix `d[,]` such that `d[i,j]` is the distance from i to j. □

Method 3.5. Dynamic programming for all-pair distances

- Initially, let `d[i,j]` be 1 (alt. the edge length of $\{i,j\}$), or ∞ if i,j are not adjacent. □
- After every step $t \geq 0$ let `d[i,j]` be the shortest length of a path between i,j such that its internal vertices are from $\{0,1,2,\ldots,t-1\}$. □
- Moving from step t to $t+1$, we update all the distances as:
  - Either `d[i,j]` from the previous step is still optimal (the vertex t does not help to obtain a shorter path),
  - or there is a shorter path through the vertex t, which is of length `d[i,t]+d[t,j]`. □

Theorem 3.6. *Method 3.5 correctly computes the distance* `d[i,j]` *between each vertex pair* i,j.

Remark: In a practical implementation we may use, say, `MAX_INT/2` in place of ∞.

Algorithm 3.7. Floyd–Warshall algorithm (cf. 3.5)

```
input < the adjacency matrix G[][] of an N-vertex graph,
    such that the vertices of G are indexed as 0...N-1,
    and G[i,j]=1 if i,j adjacent and G[i,j]=0 otherwise;

for (i=0; i<N; i++) for (j=0; j<N; j++)
    d[i,j] = (i==j?0: (G[i,j]? 1: MAX_INT/2));
for (t=0; t<N; t++) {
    for (i=0; i<N; i++) for (j=0; j<N; j++)
        d[i,j] = min(d[i,j], d[i,t]+d[t,j]);
}
return 'The distance matrix d[][]'; □
```

Notice that this Algorithm 3.7 is extremely simple and relatively fast — it runs about N^3 steps to get the whole distance matrix.

Its only problem is that all-pairs distances must be computed at the same time, even if we need to know just one distance...

3.3 Weighted distance in graphs

Definition 3.8. **A weighted graph** is a graph G together with a weighting w of the edges by real numbers $w : E(G) \to \boldsymbol{R}$ (edge lengths in this case). A *positively weighted graph* G, w is such that $w(e) > 0$ for all edges e.

The edge weights $w(e)$ are sometimes called also *edge costs*. □

Definition: Consider a positively weighted graph G, w. The length of the weighted walk $S = v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$ in G is the sum

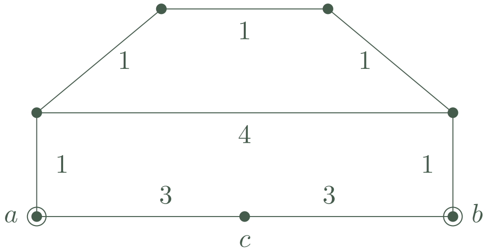
$$d_G^w(S) = w(e_1) + w(e_2) + \ldots + w(e_n)\,.$$

The *weighted distance* in G, w between a vertex pair u, v is

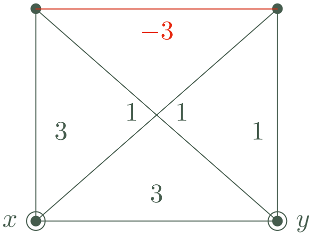
$$d_G^w(u,v) = \min\{d_G^w(S) : S \text{ is a walk between } u, v\}\,. □$$

Analogously to Section 3.1 we have:

Lemma 3.9. *The weighted distance in a positively weighted graph satisfies the triangle inequality.*



The distances between a–c and between b–c are 3. What about the a–b distance?
Is it 6? No, the distance from a to b in the graph is 5 (traverse the upper v.).



And what is the x–y distance now? Say, 3 or 1? No, it is $-\infty$.
We have got a very good reason to forbid negative edges!

3.4 Computing the shortest paths

This section with the more specific problem of finding the shortest distance between one pair of terminals in a graph. This very frequent problem is usually solved using Dijkstra's or A^* algorithms.

Remark: The coming Dijkstra's algorithm is, on one hand, slightly more involved than Algorithm 3.7, but it is significantly faster in the computation of single shortest distance, on the other hand. □

Dijkstra's algorithm

- Is a variant of graph searching (related to BFS), in which every discovered vertex carries a *variable keeping its temporary distance*—the length of the shortest so far discovered walk reaching this vertex from the starting vertex. □
- We always pick from the depository the vertex with the shortest temporary distance. This is because no shorter walk may reach this vertex (assuming nonnegative edge lengths). □
- At the end of processing, the temporary distances become final shortest distances from the starting vertex (cf. Theorem 3.12).

Algorithm 3.10. Computing the single-source shortest paths (Dijkstra)

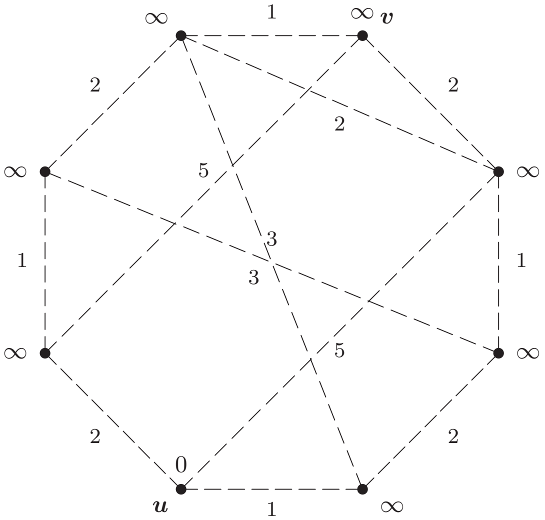
Finding the shortest path(s) from u to v, or from u to all other vertices.

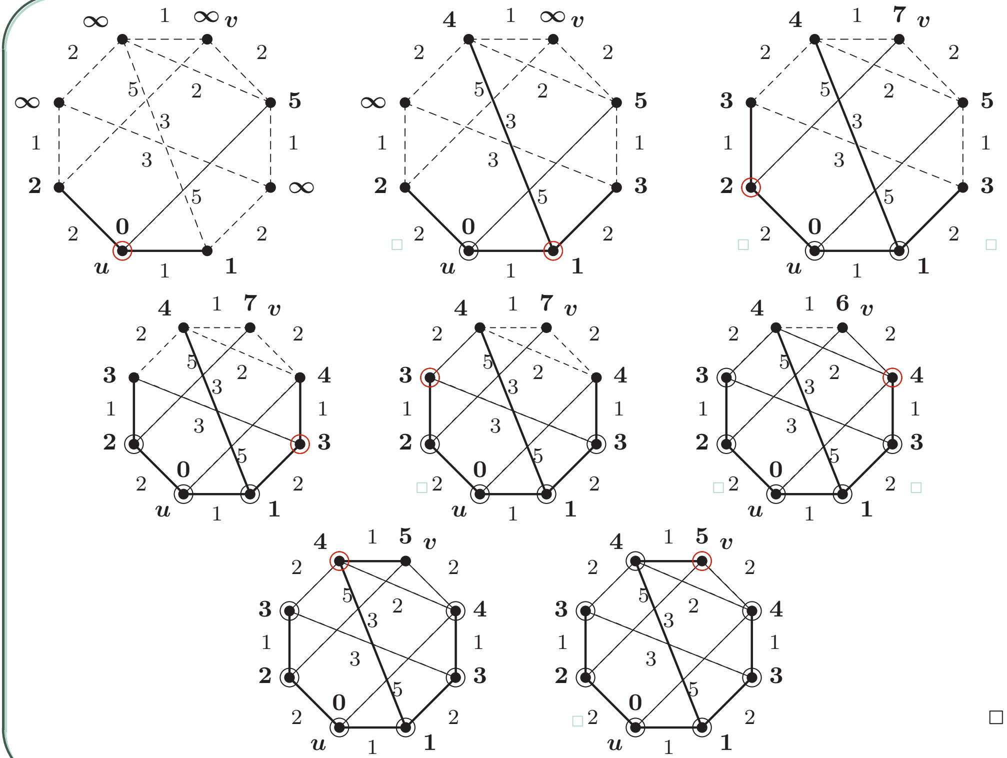
```
input < N-vertex graph given by adjacencies neib[][] and corr. lengths len[][],
    where neib[i][0],...,neib[i][dg[i]-1] are the neighbours of a
    degree-dg[i] vertex i, and the length from i to neib[i][k] is len[i][k]>0;
input < u,v, where u is the starting vertex and v the destination;

// state[i] records the vertex processing state, dist[i] is the temporary distance
for (i=0; i<=N; i++) { dist[i] = MAX_INT; state[i] = 0; }
dist[u] = 0;
while (state[v]==0) {
    for (i=0, j=N; i<N; i++)
        if (state[i]==0 && dist[i]<dist[j]) j = i;
    // picking the nearest unprocessed vertex j, and processing it...
    if (dist[j]==MAX_INT) return 'No path';
    state[j] = 1;
    for (k=0; k<dg[j]; k++)
        if (dist[j]+len[j][k]<dist[neib[j][k]]) {
            incm[neib[j][k]] = j;
            dist[neib[j][k]] = dist[j]+len[j][k];
        }
}
return 'A u-v path of length dist[v], stored in incm[] reversely';
```

Remark: Notice that Algorithm 3.10 works as-is also in directed graphs.

Example 3.11. *An illustration run of Dijkstra's Algorithm 3.10 from u to v in the following graph.*




u
v

□

Fact: The number of steps performed by Algorithm 3.10 to find the shortest path from `u` to `v` is about N^2 when N is the number of vertices (not so good...). □
On the other hand, with a better implementation of the depository, one can achieve on sparse graphs runtime almost linear in the number of edges. □

Theorem 3.12. *Every iteration of Algorithm 3.10 (since just after finishing the first* `while()` *loop) maintains an invariant that*

- `dist[i]` *is the length of a shortest path from* `u` *to* `i` *using only those internal vertices* x *of* `state[x]==1` *(finished).* □

Proof: Briefly using *mathematical induction*:

- In the first iteration, the first vertex `j=u` is picked and processed, and its neighbours receive the correct straight distances (edge lengths). □

- In every next iteration, the picked vertex `j` is the nearest one to the starting vertex `u`. Assuming nonnegative costs `del[][]`, this certifies that no shorter walk from `u` to `j` may exist in the graph. □

  On the other hand, any improved path from `u` to an unfinished vertex `i` passing through `j` has `ij` as the last edge (since the distance of `j` is not smaller than of the other finished vertices). Hence `dist[i]` is updated correctly in the algorithm.

□

In some situations, there is a better alternative to ordinary Disjktra's algorithm — the *Algorithm A^** which uses a suitable *potential function* to direct the search "towards the destination". Whenever we have a good "sense of direction" (e.g. in a topo-map navigation), A^* can perform much better!

Algoritmus A^*

- It re-implements Dijkstra with suitably modified edge costs. □
- Let $p_v(x)$ be a potential function giving an arbitrary lower bound on the distance from x to the destination v. E.g., in a map navigation, $p_v(x)$ may be the Euclidean distance from x to v. □
- Each directed(!) edge xy of the weighted graph G, w gets a new cost

  $$\boldsymbol{w'(xy) = w(xy) + p_v(y) - p_v(x)\,.}$$

  The potential p_v is *admissible* when all $w'(xy) \geq 0$, i.e. $w(xy) \geq p_v(x) - p_v(y)$.

  The above Euclidean potential is always admissible. □
- The modified length of any u-v walk S then is $d_G^{w'}(S) = d_G^w(S) + p_v(v) - p_v(u)$, which is a constant difference from $d_G^w(S)$! Hence some S is optimal for the weighting w iff S is optimal for w'.

  Here the Euclidean potential "strongly prefers" edges in the dest. direction.