$$\int \frac{x^4}{x^3-3x^2+4}\,dx = \int\left[(x+3)+\frac{9x^2-4x-12}{x^3-3x^2+4}\right]dx$$

$$x^4 : x^3-3x^2+4 = x+3$$
$$x^4-3x^3+4x$$
$$3x^3-4x$$
$$3x^3-9x^2+12$$
$$\underline{9x^2-4x-12}$$

$$\int \frac{9x^2-4x-12}{x^3-3x^2+4}\,dx = \int\left(\frac{9x^2-18x}{x^3-3x^2+4}+\frac{14x-12}{x^3-3x^2+4}\right)dx$$

	1	-3	0	4
-1	1	-4	4	0

$$\Rightarrow x^3-3x^2+4 = (x+1)(x^2-4x+4) =$$
$$= (x+1)(x-2)^2$$

Rozložíme na parc. zlomky

$$\frac{7x-6}{x^3-3x^2+4} = \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$7x-6 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

$$x=-1: \quad -13 = 9A \Rightarrow A = -\frac{13}{9}$$

$$x=2: \quad 8 = 3C \Rightarrow C = \frac{8}{3}$$

$$x^2: \quad 0 = A+B \Rightarrow B = -A = \frac{13}{9}$$

$$\int \frac{7x-6}{x^3-3x^2+4}\,dx = -\frac{13}{9}\int\frac{1}{x+1}\,dx + \frac{13}{9}\int\frac{1}{x-2}\,dx + \frac{8}{3}\int\frac{1}{(x-2)^2}\,dx =$$

$$= -\frac{13}{9}\ln(x+1) + \frac{13}{9}\ln(x-2) - \frac{8}{3}\,\frac{1}{x-2}$$

$$\int\frac{x^5}{x^3-3x^2+4} = \int(x+3) + \int\frac{9x^2-18x}{x^3-3x^2+4} + 2\int\frac{7x-6}{x^3-3x^2+4}\,dx =$$

$$\frac{x^2}{2} + 3x + 3\ln\left(x^3 - 3x^2 + 4\right) + \frac{26}{9}\ln|x-2| - \frac{26}{9}\ln|x+1| +$$

$$- \frac{16}{3}\,\frac{1}{x-2} =$$

$$= \frac{x^2}{3} + 3x + 3\ln|x+1| + 6\ln|x-2| + \frac{26}{9}\ln|x-2| - \frac{26}{9}\ln|x+1| +$$

$$- \frac{16}{3}\,\frac{1}{x-2}$$

$$= \frac{x^2}{3} + 3x + \frac{1}{9}\ln|x+1| + \frac{80}{9}\ln|x-2| - \frac{16}{3}\,\frac{1}{x-2}$$

$$\int \frac{x^2+x}{x^4+2x^3+5x^2+4x+4}\,dx$$

	1	2	5	4	4
-2	1	0	5	-6	16
-4	1	-2	13	-48	+ mnoz

Polynom nemá rac. kořeny. Budeme hledat násobné kořeny, tj. kořeny polynomu i jeho derivace, tedy kořeny n.s.d. polynomu a jeho derivace:

$$x^4+2x^3+5x^2+4x+4 : 4x^3+6x^2+10x+4$$

$$2x^4+4x^3+10x^2+8x+8 : 2x^3+3x^2+5x+2 = x+1$$

$$2x^4+3x^3+5x^2+2x$$

$$x^3+5x^2+6x+8$$

$$2x^3+10x^2+12x+16 \quad \Big| \; 2x^3+3x^2+5x+2$$

$$\boxed{7x^2+7x+14}$$

$$\left(4x^3+6x^2+10x+4\right):\underline{\left(x^2+x+2\right)} = 4x+2$$

$$4x^3+4x^2+8x$$

$$2x^2+2x+4$$

$$\Rightarrow \text{jm.} = (x^2+x+2)^2$$

$$\underline{\int \frac{x^2+x}{(x^2+x+2)^2}\,dx} = \int \frac{dx}{x^2+x+2} - 2\int \frac{dx}{(x^2+x+2)^2}$$

a) $$\frac{x^2+x}{(x^2+x+2)^2} = \frac{Ax+B}{(x^2+x+2)} + \frac{Cx+D}{(x^2+x+2)^2} = \frac{1}{x^2+x+2} - \frac{2}{(x^2+x+2)^2}$$

$$x^2+x = (Ax+B)(x^2+x+2) + Cx+D = B(x^2+x+2)+Cx+D$$

$x^3: 0 = A$

$x^2: 1 = B$

$x: 1 = B + C \Rightarrow C = 0 \Rightarrow x^0: 0 = 2B + D \Rightarrow D = -2$

$$\int \frac{dx}{(x^2+x+2)} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}} = \frac{4}{7}\int \frac{dx}{\left(\frac{2x+1}{\sqrt{7}}\right)^2+1} =$$

$$\left[\begin{array}{l} t = \frac{2x+1}{\sqrt{7}} \\ dt = \frac{2}{\sqrt{7}}\,dx \end{array}\right]$$

$$= \frac{4}{7}\frac{\sqrt{7}}{2}\int \frac{dt}{t^2+1} = \frac{2}{\sqrt{7}}\,\mathrm{arctg}\, t = \frac{2}{\sqrt{7}}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{7}}\right)$$

$$\int \frac{dx}{(x^2+x+2)^2} = \int \frac{dx}{\left(\left(x+\frac{1}{2}\right)^2+\frac{7}{4}\right)^2} = \frac{16}{49}\int \frac{dx}{\left(\left(\frac{2x+1}{\sqrt{7}}\right)^2+1\right)^2} =$$

$$\left[\begin{array}{l} \operatorname{tg} t = \frac{2x+1}{\sqrt{7}} \\ \frac{dt}{\cos^2 t} = \frac{2}{\sqrt{7}}\,dx \end{array}\right]$$

$$= \frac{16}{49}\cdot\frac{\sqrt{7}}{2}\int \frac{\frac{1}{\cos^2 t}}{(\operatorname{tg}^2 t+1)^2}\,dt = \frac{8}{7\sqrt{7}}\int \frac{\frac{1}{\cos^2 t}}{\left(\frac{1}{\cos^2 t}\right)^2}\,dt =$$

$$\left[\begin{array}{l} \operatorname{tg}^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} \\ \cos^2 x\,\operatorname{tg}^2 x = 1-\cos^2 x \\ \cos^2 x = \frac{1}{1+\operatorname{tg}^2 x} \end{array}\right] \qquad \sin^2 x = \frac{\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x}$$

$$= \frac{8}{7\sqrt{7}}\int \cos^2 t\,dt = \frac{8}{7\sqrt{7}}\cdot\frac{1}{2}\left(t+\sin t\cdot\cos t\right) =$$

$$= \frac{4}{7\sqrt{7}}\left(\operatorname{arctg}\left(\frac{2x+1}{\sqrt{7}}\right) + \frac{\frac{2x+1}{\sqrt{7}}}{\frac{1}{7}(4x^2+4x+8)}\right) = \frac{4}{7\sqrt{7}}\left[\operatorname{arctg}\left(\frac{2x+1}{\sqrt{7}}\right) + \sqrt{7}\cdot\frac{2x+1}{4x^2+4x+8}\right]$$

$$[f(t), g(t)]$$

$$f'(t) = 2t$$

$$g'(t) = t^2$$

$$\int_0^4 \sqrt{4t^2 + t^4}\, dt =$$

$$= \int_0^4 t\sqrt{4 + t^2}\, dt =$$

$$= \frac{1}{3}\left[(4 + t^2)^{\frac{3}{2}}\right]_0^4 = \frac{1}{3}\left(20^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$$

$$\int_1^2 \sqrt{\frac{1}{x-1}}\, dx = \lim_{\delta \to 1_+} \int_\delta^2 \sqrt{\frac{1}{x-1}} = \lim_{\delta \to 1_+} \left[2\sqrt{x-1}\right]_\delta^2 =$$

$$= \lim_{\delta \to 1_+} 2 - 2\sqrt{\delta - 1} = 2$$

$$\int_0^\infty e^{-x} \cos(x)\, dx = \lim_{\delta \to \infty} \int_0^\delta e^{-x} \cos(x)\, dx = \frac{1}{2} \lim_{\delta \to \infty} \left[e^{-x} \sin x + e^{-x} \cos x\right]_0^\delta$$

$$= \frac{1}{2}$$

$$f(x) = \ln(x)$$
$$f'(x) = \frac{1}{x}$$
$$f''(x) = -\frac{1}{x^2}$$
$$f^{(3)}(x) = 2\,\frac{1}{x^3}$$
$$f^{(4)}(x) = -6 \cdot \frac{1}{x^4}$$
$$\vdots$$
$$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,\frac{1}{x^n}$$

$$f(2) = \ln(2)$$
$$f'(2) = \frac{1}{2}$$
$$f^{(2)}(2) = -\frac{1}{2^2}$$
$$\vdots$$
$$f^{(n)}(2) = (-1)^{n-1}(n-1)!\,\frac{1}{2^n}$$

$$\ln(x) = \ln(2) + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \dots + (-1)^{(n-1)}\frac{1}{n\,2^n}(x-2)^n$$

$$= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n}(x-2)^n$$

$$a_{n+1} = \frac{(-1)^n}{(n+1)2^{n+1}}$$
$$a_n = \frac{(-1)^{n-1}}{n\,2^n}$$

poloměr konvergence: $R: \dfrac{1}{\lim\limits_{n\to\infty}\left|\frac{n}{2(n+1)}\right|} = 2$

řada konverguje pro $(x-2) \in \langle -2, 2)$ $\Leftrightarrow$ $x \in \langle 0, 4)$

pro $x = 0$: $\ln(2) - \frac{1}{1} - \frac{1}{2} - \frac{1}{3} \dots$ diverguje

$x = 4$: $\ln(2) + \underbrace{1 - \frac{1}{2} + \frac{1}{3}} \dots$ konverguje

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Sečtěte řadu $\sum_{n=1}^{\infty} \frac{1}{n 2^n}$

$$\int_2^{\infty} \frac{dx}{x^{n+1}} = \lim_{\delta \to \infty} \int_2^{\delta} \frac{dx}{x^{n+1}} = \lim_{\delta \to \infty} \left[-\frac{1}{n x^n} \right]_2^{\delta} =$$

$$= \lim_{\delta \to \infty} \left[-\frac{1}{n \delta^n} + \frac{1}{n 2^n} \right] = \frac{1}{n 2^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n 2^n} = \sum_{n=1}^{\infty} \int_2^{\infty} \frac{dx}{x^{n+1}} \underline{=} \int_2^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{x^{n+1}} \right) dx$$

konverguje stejnoměrně na $\langle 2, \infty)$

Hodnoty fcí $\frac{1}{x^{n+1}}$ na int $(2,\infty)$ lze odhadnout výrazem $\frac{1}{2^{n+1}}$, ale $\sum_{n=1}^{\infty}\frac{1}{2^{n+1}}$ konverguje, tedy $\sum_{n=1}^{\infty}\frac{1}{x^{n+1}}$ konverguje stejnoměrně na $(2,\infty)$

$$\int_2^{\infty}\sum_{n=1}^{\infty}\frac{1}{x^{n+1}} = \int_2^{\infty}\frac{1}{x^2}\left(\sum_{n=0}^{\infty}\frac{1}{x^n}\right) = \int_2^{\infty}\frac{1}{x^2}\left(\frac{1}{1-\frac{1}{x}}\right)dx =$$

$$= \int_2^{\infty}\frac{1}{x(x-1)}\,dx = \lim_{\delta\to\infty}\int_2^{\delta}\left(\frac{1}{x-1}-\frac{1}{x}\right)dx =$$

$$= \lim_{\delta\to\infty}\Big[\ln(x-1)-\ln(x)\Big]_2^{\delta} =$$

$$\lim_{\delta\to\infty}\Big(\underbrace{\ln(\delta-1)-\ln(\delta)}_{\ln\left(\frac{\delta-1}{\delta}\right)}\Big)+\ln(2) = \ln(2)$$