Najdeme nejdříve ortogonální bázi f_1, f_2:

$f_1 := \frac{1}{x}$

Volíme $f_2 = \ell \cdot \frac{1}{x} + \frac{1}{x^2}$, kde $\ell \in \mathbb{R}$.

Chceme $\langle f_1, f_2 \rangle = 0$, tedy

$\langle\ \rangle$ ... skal. součin

$$0 = \langle f_1, f_2 \rangle = \left\langle f_1, \ell \cdot \frac{1}{x} + \frac{1}{x^2} \right\rangle = \ell \left\langle f_1, \frac{1}{x} \right\rangle + \left\langle f_1, \frac{1}{x^2} \right\rangle =$$

$$\Rightarrow \ell = -\frac{\langle f_1, \frac{1}{x^2} \rangle}{\langle f_1, \frac{1}{x} \rangle} = -\frac{\langle \frac{1}{x}, \frac{1}{x^2} \rangle}{\langle \frac{1}{x}, \frac{1}{x} \rangle}$$

$$\left\langle \frac{1}{x}, \frac{1}{x^2} \right\rangle = \int_1^2 \frac{1}{x} \cdot \frac{1}{x^2}\, dx = \int_1^2 \frac{1}{x^3} = \left[-\frac{1}{2x^2} \right]_1^2 = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

$$\left\langle \frac{1}{x}, \frac{1}{x} \right\rangle = \int_1^2 \frac{1}{x^2}\, dx = \left[-\frac{1}{x} \right]_1^2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \ell = -\frac{3}{4}$$

$$f_1 = \frac{1}{x}$$

$$f_2 = \frac{1}{x^2} - \frac{3}{4}\frac{1}{x}$$

$$\left(\langle f_1, f_2 \rangle = 0 = \left(\int_1^2 \frac{1}{x}\left(\frac{1}{x^2} - \frac{3}{4}\frac{1}{x}\right) dx \right) \right.$$

$$\| f_1 \| = \sqrt{\langle f_1, f_1 \rangle} = \frac{1}{\sqrt{2}}$$

$$\langle f_2, f_2 \rangle = \int_1^2 \left(\frac{1}{x^2} - \frac{3}{4}\frac{1}{x} \right)^2 dx = \int_1^2 \left(\frac{1}{x^4} - \frac{3}{2}\cdot\frac{1}{x^3} + \frac{9}{16}\frac{1}{x^2} \right) dx =$$

$$= \left[-\frac{1}{3x^3} \right]_1^2 - \frac{3}{2}\cdot\frac{3}{8} + \frac{9}{16}\cdot\frac{1}{2} = \left(\frac{1}{3} - \frac{1}{24} \right) - \frac{9}{16} + \frac{9}{32} =$$

$$= \frac{7}{24} - \frac{9}{32} = \frac{28 - 27}{96} = \frac{1}{96}$$

$$\| f_2 \| = \sqrt{\langle f_2, f_2 \rangle} = \frac{1}{4\sqrt{6}}$$

$:$

$f.$

Ortonormální báze:

$$e_1 = \frac{f_1}{\|f_1\|} = \frac{\sqrt{2}}{x}$$

$$e_2 = \frac{f_2}{\|f_2\|} = 4\sqrt{6}\cdot\left(\frac{1}{x^2} - \frac{3}{4x}\right) = \frac{4\sqrt{6}}{x^2} - \frac{3\sqrt{6}}{x}$$

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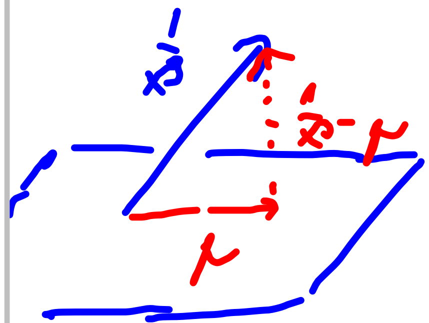
2) Spočítáme kolmý průmět $\frac{1}{x^3}$ do prostoru generovaného vektory $\frac{1}{x}, \frac{1}{x^2}$ (tj. vektory f_1, f_2):

Označme kolmý průmět jako p:

$$p = \frac{\langle \frac{1}{x^3}, f_1\rangle}{\langle f_1, f_1\rangle}\cdot f_1 + \frac{\langle \frac{1}{x^3}, f_2\rangle}{\langle f_2, f_2\rangle}\cdot f_2$$

$$\left\langle \frac{1}{x^3}, f_1\right\rangle = \int_1^2 \frac{1}{x^3}\,dx = \frac{7}{24}$$

$$\left\langle \frac{1}{x^3}, f_2 \right\rangle = \int_1^2 \left(\frac{1}{x^5} - \frac{3}{4x^4} \right) dx =$$



$$= \left[-\frac{1}{4x^4} \right]_1^2 - \frac{3}{4} \cdot \frac{7}{24} =$$

$$= \frac{1}{4} - \frac{1}{64} - \frac{7}{32} = \frac{16-1-14}{64} = \frac{1}{64}$$

celkem

$$p = \frac{\frac{7}{24}}{\frac{1}{2}} \cdot f_1 + \frac{\frac{1}{64}}{\frac{1}{96}} \cdot f_2 = \frac{7}{12} \cdot \frac{1}{x} + \frac{3}{2}\left(\frac{1}{x^2} - \frac{3}{4x} \right) =$$

$$= \frac{3}{2x^2} - \frac{13}{24x}$$

Vzdálenost $\frac{1}{x^3}$ od $\left\langle \frac{1}{x}, \frac{1}{x^2} \right\rangle$

$$\left\| \frac{1}{x^3} - p \right\| = \left\| \frac{1}{x^3} - \frac{3}{2x^2} + \frac{13}{24x} \right\| = \frac{1}{24\sqrt{10}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin(nx) + b_n \cos(nx)$$

promítáme $f(x)$ do $\langle 1, \sin x, \cos x, \sin 2x, \cos 2x, \dots \rangle$

$$a_n = \frac{\langle f(x), \sin(nx) \rangle}{\langle \sin(nx), \sin(nx) \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin(nx)\, dx$$

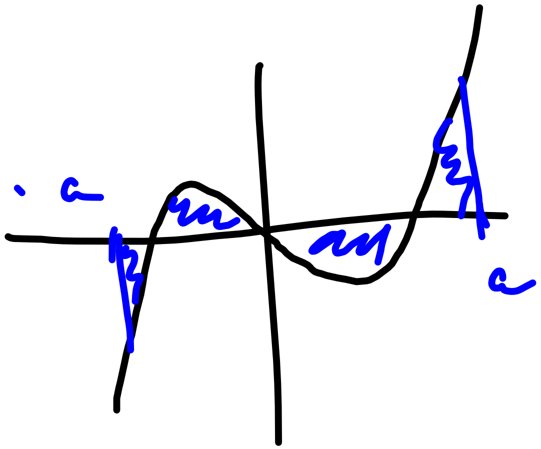
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(nx)\, dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$f(x) = x$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x = 0$$

(integrál z liché fce na intervalu $-a, a$ je vždy nulový)



$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{x}_{\text{lichá}} \cdot \underbrace{\cos(nx)}_{\text{sudá}}}_{\text{lichá}}\, dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \sin(nx)$$

$$\int x \sin(nx)\,dx = -\frac{1}{n} x \cos(nx) + \int \frac{1}{n} \cos(nx)\,dx =$$

$$\begin{bmatrix} u = x & u' = 1 \\ v' = \sin(nx) & v = -\frac{1}{n}\cos(nx) \end{bmatrix}$$

$$= -\frac{1}{n} x \cos(nx) + \frac{1}{n^2} \sin(nx) + C$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot \sin(nx)\,dx = \frac{1}{\pi} \left[-\frac{1}{n} x \cos(nx) + \frac{1}{n^2} \sin(nx) \right]_{-\pi}^{\pi} =$$

$$= \frac{1}{\pi} \left(-\frac{1}{n} \pi (-1)^n + \frac{1}{n} (-\pi) \cdot (-1)^n \right) =$$

$$= \left(-\frac{1}{n} (-1)^n + \frac{1}{n} (-1)^n \right) = \frac{2 \cdot (-1)^{n+1}}{n}$$

$$x = \sum_{n=1}^{\infty} a_n \sin(nx) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

$$x + 1 = 1 + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

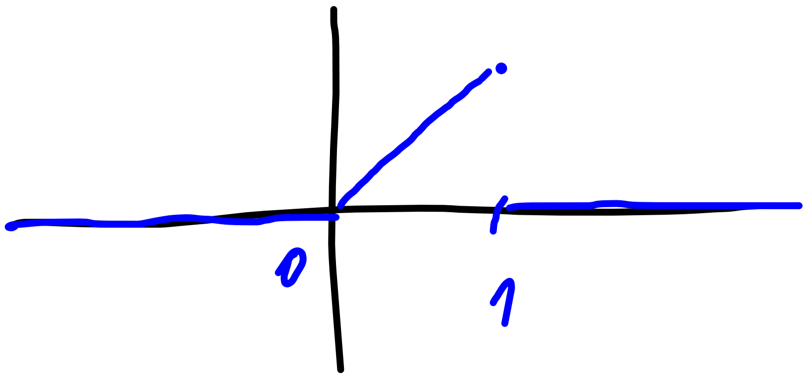
Při rozvíjení funkce $\frac{1}{x}$ by bylo třeba počítat

$$\int_{-\pi}^{\pi} \frac{1}{x} \cdot \sin(nx)\,dx \quad \ldots \quad \text{numericky}$$

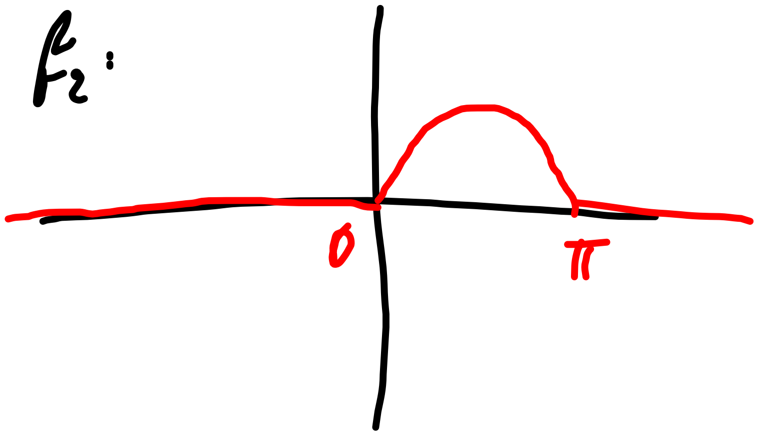
$$f_1 * f_2(t) = \int_{-\infty}^{\infty} f_1(t-x)\, f_2(x)\, dx$$

$$\left(= \int_{-\infty}^{\infty} f_2(t-x)\, f_1(x)\, dx \right)$$

f_1



f_2:



$$f_1 * f_2(t) = \int_{-\infty}^{\infty} f_1(t-x)\, f_2(x)\, dx$$

integraci stačí provést přes interval, kde $f_1(t-x)\cdot f_2(x)$ je nenulová. Funkce $f_1(t-x)\cdot f_2(x)$ uvažovaná jako funkce jedné proměnné x s parametrem t není nulová funkcí pouze pro některá t:

$$f_1(t-x)\cdot f_2(x) \neq 0 \iff f_1(t-x) \neq 0 \;\&\; f_2(x) \neq 0 \iff$$

$$\iff t-x \in \langle 0, 1\rangle \;,\; x \in (0, \pi)$$

$$\Updownarrow$$

$$x \in \langle t-1, t\rangle$$

funkce $f_1(t-x)\cdot f_2(x)$ nebude konstantní nulovou fcí pro $\langle t-1, t\rangle \cap (0, \pi) \neq \emptyset$

$\Rightarrow \ell \in (0, \pi + 1)$

Budeme počítat

$$\int f_1(\ell - x) \cdot f_2 \, dx = \int (\ell - x) \cdot \sin x \, dx$$

$$= \ell \int \sin x - \underset{\substack{\shortparallel \\ \left(-x\cos x + \underset{\substack{\shortparallel \\ -x\cos x + \sin x}}{\int \cos x} \right)}}{\int x \sin x} = -\ell \cos x + x \cos x - \sin x$$

$$\underbrace{\int_{-\infty}^{\infty} f_1(t-x)\, f_2(x)\, dx}_{*} = \int_{\langle t-1,\, t\rangle \cap \langle 0,\, \pi\rangle} f_1(t-x)\, f_2(x)\, dx :$$

a) $t \in (0,1) \Rightarrow \langle t-1,\, t\rangle \cap \langle 0,\, \pi\rangle = \langle 0,\, t\rangle$

$$* = \int_0^t (t-x)\sin x\, dx = \left[-t\cos x + x\cos x - \sin x\right]_0^t =$$

$$= -t\cos t + t\cos t - \sin t + t = t - \sin t$$

b) $t \in (1,\, \pi)$:

$$\int_{t-1}^{t} (t-x)\sin x\, dx = \left[-t\cos x + x\cos x - \sin x\right]_{t-1}^{t} =$$

$$= \sin(t-1) - \sin t + \cos(t-1)$$

c) $t \in [\pi, \pi+1)$

$$* = \int_{t-1}^{\pi} (t-x)\cdot \sin(x)\,dx = \left[-t\cos x + x\cos x - \sin x\right]_{t-1}^{\pi} =$$

$$= t - \pi + \sin(t-1) + \cos(t-1)$$

Celkem

$$f_1 * f_2 = \begin{cases} t - \sin(t) & \text{pro } t \in [0,1] \\ \sin(t-1) - \sin(t) + \cos(t-1) & \text{pro } t \in [1,\pi] \\ t - \pi + \sin(t-1) + \cos(t-1) & \text{pro } t \in [\pi, \pi+1) \\ 0 & \text{jinak} \end{cases}$$