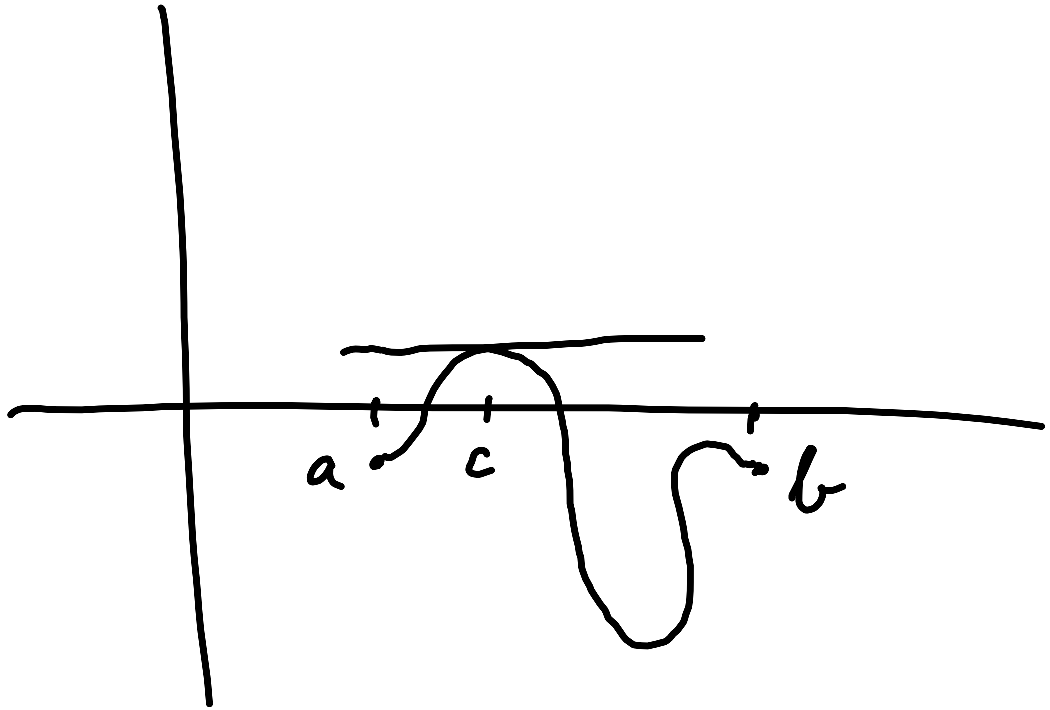
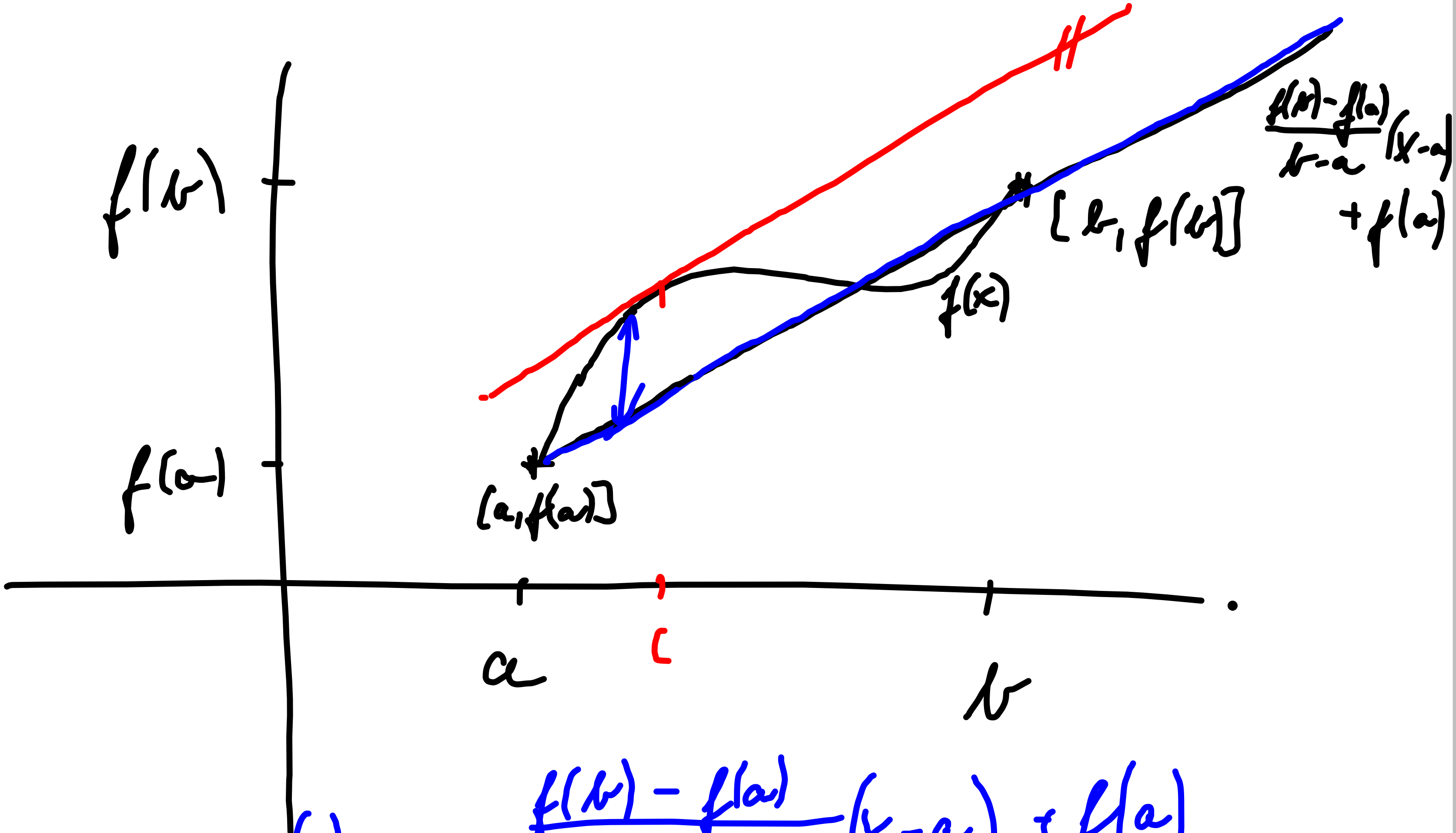

a
c
b



$$g(x) = y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$$

$$h(x) := f(x) - g(x), \quad h(b) = h(a) = 0$$

Tedy na $h(x)$ použijeme Rolleovu větu:

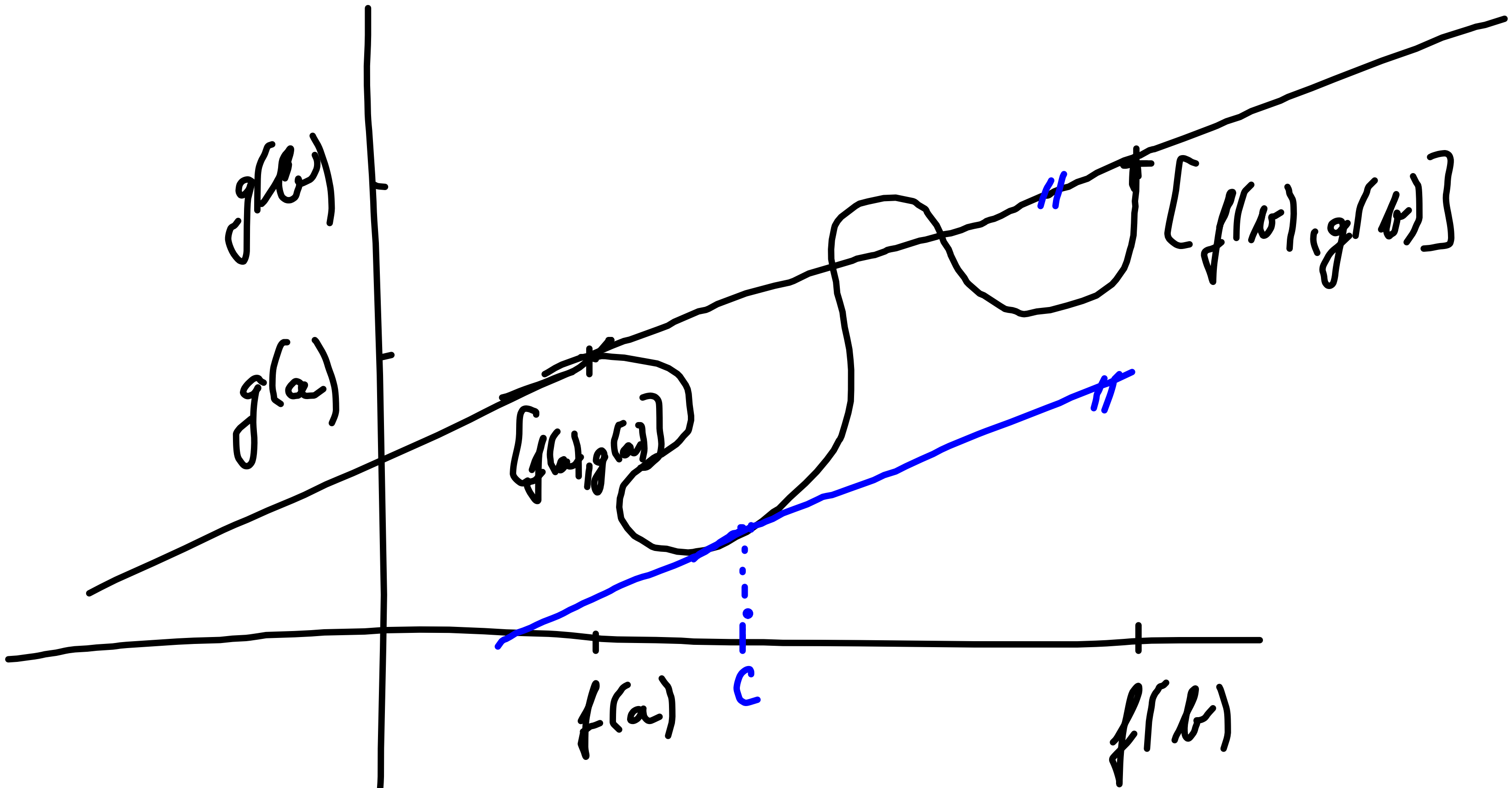
ex. $c \in (a, b)$: $h'(c) = 0$

$$h'(x) = \left\{ f(x) - \left[\frac{f(b) - f(a)}{b-a}(x-a) + f(a) \right] \right\}' =$$

$$= f'(x) - \frac{f(b) - f(a)}{b-a}$$

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b-a}$$

$$\Leftrightarrow \quad f'(c) = \frac{f(b) - f(a)}{b-a}$$



$$h(t) = (f(b) - f(a))\,g(t) - (g(b) - g(a))\,f(t)$$

$$h(a) = f(b)\,g(a) - g(b)\,f(a)$$

$$h(b) = -f(a)\,g(b) + f(b)\,g(a) = h(a)$$

h opět splňuje podmínky Rolleovy věty, tedy

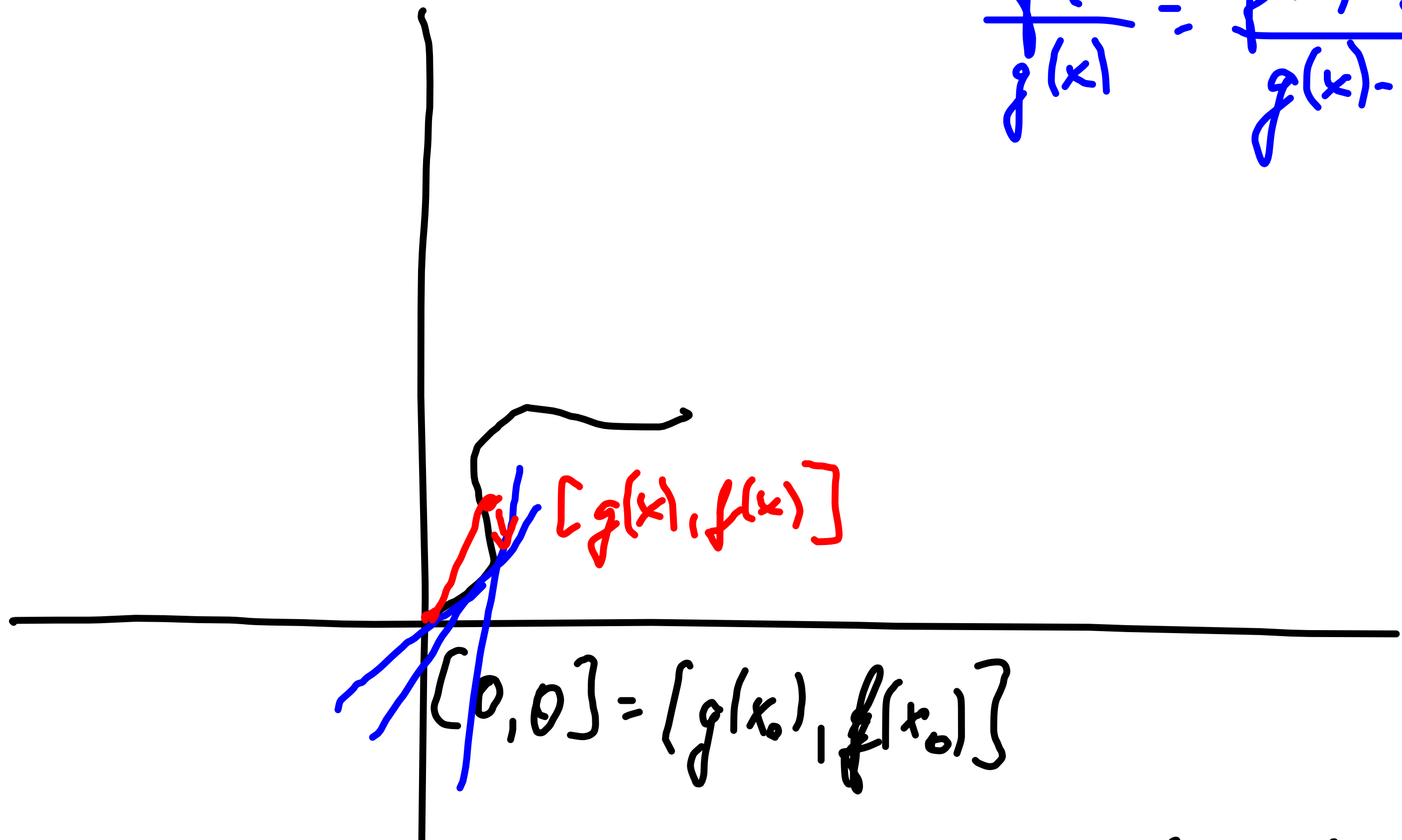
$\exists c \in (a,b)$: $h'(c) = 0$, neboli

$$0 = (f(b)-f(a))g'(c) - (g(b)-g(a))f'(c)$$

$g'(c) \neq 0$ $\Leftrightarrow$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)}$$



Aplikujme předchozí formulaci vety o střední hodnotě na interval (x_0, x):

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c_x)}{g'(c_x)}$$

Předpokládáme, že ex. okolí bodu x_0 takové, že g' je na něm nenulová.

Předpokládáme dále, že $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = C$ existuje,

tj. pro lib posloupnost $y_n \to x_0$ platí, že

$$\lim_{n \to \infty} \frac{f'(y_n)}{g'(y_n)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Vezměme tedy lib. $x_n \to x_0$, potom existuje posloupnost $c_{x_n} \to x_0$ taková, že $\frac{f'(c_{x_n})}{g'(c_{x_n})} = \frac{f(x_n)}{g(x_n)}$

Tzn. že

$$\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = \lim_{n \to \infty} \frac{f'(c_{x_n})}{g'(c_{x_n})} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = C$$

$$\Rightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = C = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

$$\lim_{x\to\infty} \sqrt[x]{x} = \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x} =$$

$$= e^{\lim\limits_{x\to\infty} \frac{1}{x}\ln x} = e^{0} = 1$$

$$\lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$$\lim_{x\to 0_+} (x\cdot \ln x) = \lim_{x\to 0_+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0_+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0_+}(-x) = 0$$

$$f(x) = x\cdot \sin(x^{-1})\, e^{-\frac{1}{x^2}}$$

$$g(x) = e^{-\frac{1}{x^2}}$$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} x\cdot \sin(x^{-1}) = 0$$

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} =$$

$$\lim_{x\to 0} \frac{\sin(x^{-4})\,e^{-\frac{1}{x^2}} - 4x^{-4}\cos(x^{-4})\,e^{-\frac{1}{x^2}} + \frac{2}{x^2}\sin(x^{-1})\,e^{-\frac{1}{x^2}}}{\frac{2}{x^3}\cdot e^{-\frac{1}{x^2}}} =$$

$$= -\infty$$

$$\lim_{r \to 0} \ln\left(\left(\frac{x_1^r + \dots + x_n^r}{n}\right)^{\frac{1}{r}}\right) =$$

$$= \lim_{r \to 0} \frac{1}{r} \ln\left(\frac{x_1^r + \dots + x_n^r}{n}\right) =$$

$$= \lim_{r \to 0} \frac{\frac{x_1^r \ln x_1 + \dots + x_n^r \ln x_n}{n}}{\frac{x_1^r + \dots + x_n^r}{n}} =$$

$$= \ln x_1 + \dots + \ln x_n \qquad n$$

$$(x_1^r)' =$$

$$= \left(e^{r \ln x_1}\right)' =$$

$$= \ln x_1 \, e^{r \ln x_1} =$$

$$= \ln x_1 \, x_1^r$$

První nenulová derivace v kritickém bodě je lichá ⇒ není extrém
je sudá ⇒ je extrém

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$f'(0) = a_1$$

$$f''(x) = 2a_2 + 6a_3 x + 12 a_4 x^2 + \dots$$

$$f''(0) = 2a_2$$

$$f^{(n)}(0) = n!\, a_n \Rightarrow a_n = \frac{f^{(n)}(0)}{n!}$$

$$P_k f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \dots$$

$$\frac{(x-a)^2 f^{(2)}(c)}{k!}$$