Příklad 77. *Řešte rekurenci*

$$a_n = \frac{3}{2}a_{n-1} - \frac{1}{2}a_{n-3}, a_0 = 0, a_1 = 1, a_2 = 13/2.$$

$\underline{n \geq 3}$

I.

$$a_n = \frac{3}{2}a_{n-1} - \frac{1}{2}a_{n-3} + [n=1] + 5[n=2]$$

platí $\forall n \in \mathbb{N}_0$.

II $A(x) = \sum_{n \in \mathbb{N}_0} a_n x^n$

$$A(x) = \frac{3}{2}\sum_n a_{n-1}x^n - \frac{1}{2}\sum_n a_{n-3}x^n + \sum_n \big([n=1]+5[n=2]\big)x^n$$

$$A(x) = \frac{3}{2}xA(x) - \frac{1}{2}x^3A(x) + x + 5x^2$$

$$A(x)\left[1 - \frac{3}{2}x + \frac{1}{2}x^3\right] = x + 5x^2$$

$$A(x) = 2\cdot\frac{x+5x^2}{2-3x+x^3}$$

III. rozklad na parc. zlomky

$$x^3 - 3x + 2 = (x-1)^2(x+2)$$

$$\begin{array}{c|cccc}
 & 1 & 0 & -3 & 2 \\
\hline
1 & 1 & 1 & -2 & 0 \\
\hline
1 & 1 & 2 & 0 &
\end{array}$$

$$\frac{x+5x^2}{2-3x+x^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$x+5x^2 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Roznásobením a porovnáním koeficientů u x^0, x^1, x^2

nebo dosazením

$x=1$: $6 = 3B \Rightarrow B = 2$

$x=-2$: $18 = C \cdot 9 \Rightarrow C = 2$

$x=0$: $0 = -2A + 2B + C$
$\Rightarrow A = 3$

$$A(x) = 2 \cdot \left(\frac{3}{x-1} + \frac{2}{(x-1)^2} + \frac{2}{x+2} \right)$$

IV. rozvinení zlomků do mocninné řady např. pomocí zobecněné věty:

$$\frac{1}{(1-x)^n} = \binom{n-1}{n-1} + \binom{n}{n-1}x + \binom{n+1}{n-1}x^2 + \dots$$

upravíme $A(x)$:

$$A(x) = 2\left(\frac{-3}{1-x} + \frac{2}{(1-x)^2} + \frac{1}{\underbrace{1+\frac{x}{2}}}\right)$$

$$1-\left(-\frac{x}{2}\right) = 1-\left(-\frac{1}{2}\right)\cdot x$$

$$A(x) = 2\cdot\left(-3\sum_{n\geq 0} x^n + 2\sum_{n\geq 0}(n+1)x^n + \sum\left(-\frac{1}{2}\right)^n x^n\right) \Rightarrow$$

$$\underline{\underline{a_n}} = -6 + 4(n+1) + 2\left(-\frac{1}{2}\right)^n = \underline{\underline{4n-2+2\left(-\frac{1}{2}\right)^n}}$$

POZN.: odvozený vztah lze snadno dokázat mat. indukcí!

I. platí pro $n=0,1,2$.

II. platí-li pro $n-3, n-2, n-1 \Rightarrow$ platí pro n.

$$? = \frac{3}{2}\left(4(n-1)-2+2\left(-\frac{1}{2}\right)^{n-1}\right) - \frac{1}{2}\left(4(n-3)-2+2\left(-\frac{1}{2}\right)^{n-3}\right) =$$

$$= 4n-3+3\left(-\frac{1}{2}\right)^{n-1} + 1 - \left(-\frac{1}{2}\right)^{n-3} = 4n-2+\left(-\frac{1}{2}\right)^n\left[-6+8\right]$$

Příklad 78. *S využitím vytvořující funkce pro Fibonacciho posloupnost $F(x) = x/(1 - x - x^2)$ určete vytvořující funkci "poloviční" Fibonacciho posloupnosti $(F_0, F_2, F_4, \ldots)$.*

F_n: $0, 1, 1, 2, 3, 5, 8, 13$

F_{2n}: $0, 1, 3, 8, \ldots$

$$(F_0, F_1, F_2, F_3, F_4, \ldots)$$

$$* \; \downarrow$$

$$(F_0, 0, F_2, 0, F_4, \ldots)$$

$$x^2 \leftarrow x \; \downarrow$$

$$(F_0, F_2, F_4, \ldots)$$

$$x \leftarrow -x \quad * : \; (F_0, -F_1, F_2, -F_3, F_4, \ldots)$$

$$F(-x) \xleftrightarrow{\text{v.f.p}} (F_0, -F_1, F_2, -F_3, \ldots)$$

$$\frac{1}{2}\big(F(x) + F(-x)\big) \xleftrightarrow{\text{v.f.p}} (F_0, 0, F_2, 0, \ldots)$$

$$= \frac{1}{2}\left(\frac{x}{1-x-x^2} + \frac{(-x)}{1+x-x^2}\right) = \frac{1}{2}\,\frac{x+x^2-x^3-x+x^2+x^3}{x^4-3x^2+1} = \frac{x^2}{1-3x^2+x^4}$$

$$\boxed{z = x^2} \qquad \boxed{\frac{z}{1-3z+z^2}} \xleftrightarrow{\text{v.f.p.}} (F_0, F_2, F_4, F_6, \ldots)$$

Příklad 79. *Řešte rekurenci*

$$g_n = n g_{n-1},\ g_0 = 1.$$

I. $g_n = n \cdot g_{n-1} + [n=0]$

II. $G(x) = \sum_{n\geq 0} n \cdot g_{n-1} x^n + 1$

$G(x) = \sum_{n\geq 0} g_n x^n$

$G'(x) = \sum_{n\geq 1} n g_n x^{n-1}$

$= \sum_{n\geq 0} (n+1) g_{n+1} x^n$

$= \frac{1}{x^2} \sum_{n\geq 0} (n+1) g_{n+1} x^{n+2} =$

$= \frac{1}{x^2} \left(\sum_{n\geq 0} (n+2) g_{n+1} x^{n+2} - \sum_{n\geq 0} g_{n+1} x^{n+2} \right)$

$= \frac{1}{x^2} \cdot \left(\sum_{n\geq 0} n g_{n-1} x^n - x \right) - x(G(x)-1)$

$G(x) = x^2 G'(x) + x + x(G(x) - 1)$

$G(x)[1-x] - x^2 G'(x) = 0$

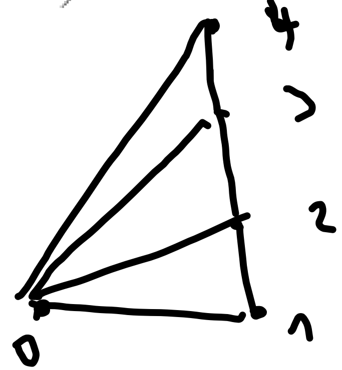
$\frac{G'(x)}{G(x)} = \frac{1-x}{x^2}$ $\int$

$\ln G(x) = \int \frac{1-x}{x^2} dx$

$G(x) = e^{\cdots}$

Příklad 80. *Vějířem řádu n nazveme graf n $n+1$ vrcholech $0, 1, \ldots, n$, který má následujících $2n-1$ hran: vrchol 0 je spojen hranou s každým ze zbylách vrcholů a každý vrchol k je spojen hranou s vrcholem $k+1$ (pro $1 \leq k < n$). Kolik koster má takový graf?*

<u>$n=4$:</u>



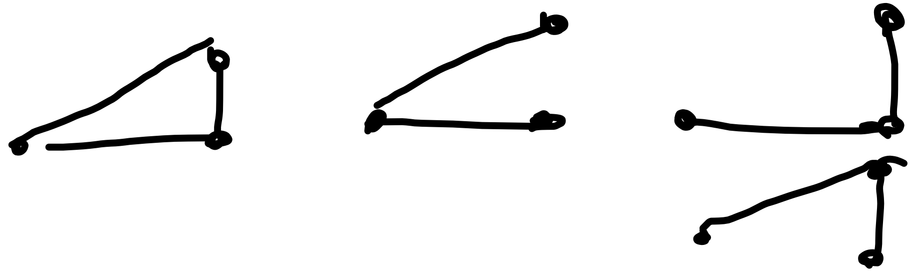
koster ?

f_n

<u>$n=1$</u>

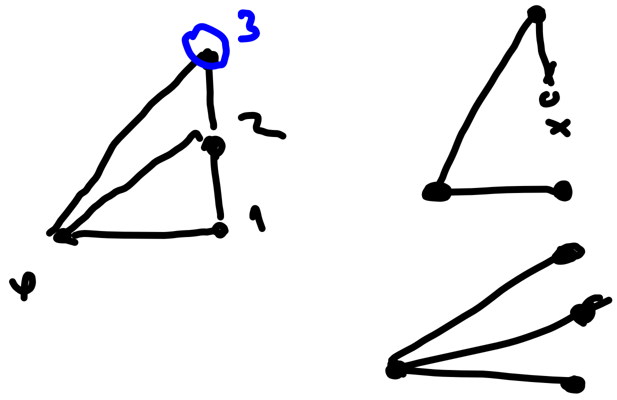
$f_1 = 1$

<u>$n=2$</u>

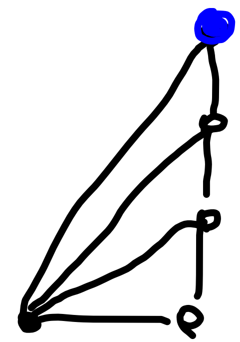


$f_2 = 3$

$n=3$



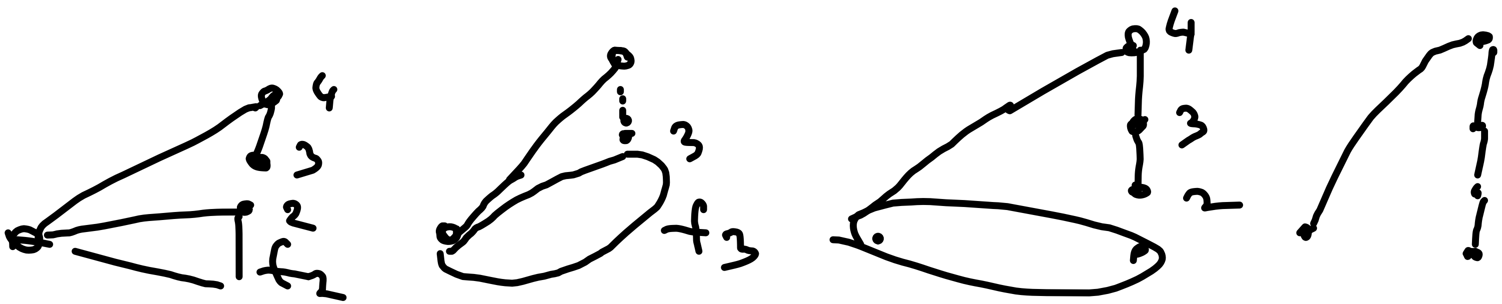
$\dots \quad f_3 = 8$



$n=4$

1. $(4,0) \notin E$

2. $(4,0) \in E$



$$f_4 = f_3 + f_3 + f_2 + f_1 + 1$$

$$f_n = f_{n-1} + f_{n-1} + f_{n-2} + \dots + f_1 + 1$$

$$f_n = f_{n-1} + \sum_{k<n} f_k + \left(1 - [n=0]\right) \qquad f_0 = 0$$

$$E(x) = \sum_n f_n x^n$$

$$E(x) = \underline{xE(x)} + \sum_{n\ge 0}\Big(\sum_{k<n} f_k\Big)x^n + \underline{x\sum_{n\ge 0} x^n}$$

$$E(x) = x \cdot E(x) + \frac{x}{1-x} + \underbrace{\sum_{n\ge 0}\sum_{k<n} f_k x^n}$$

$$\sum_{k\ge 0}\sum_{n} [n>k] f_k x^n =$$

$$= \underbrace{\sum_{k\ge 0} f_k x^k}_{E(x)} \cdot \underbrace{\sum_{n\ge 0} [n>k]\, x^{n-k}}_{\frac{x}{1-x}}$$

$$E(x) = x \cdot E(x) + \frac{x}{1-x} + E(x) \cdot \frac{x}{1-x}$$

$$E(x)\left(1 - x - \frac{x}{1-x}\right) = \frac{x}{1-x} \qquad \Big/ \cdot (1-x)$$

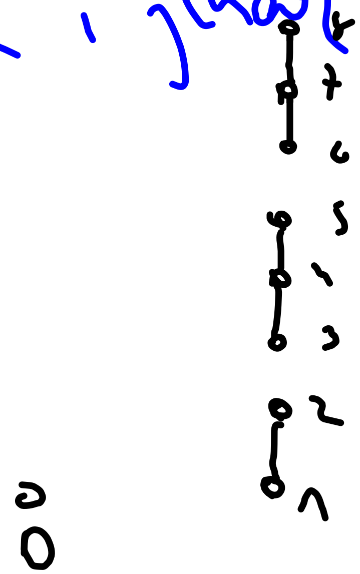
$$E(x)\left[(1-x)^2 - x\right] = x$$

$$E(x) = \frac{x}{1-3x+x^2}$$

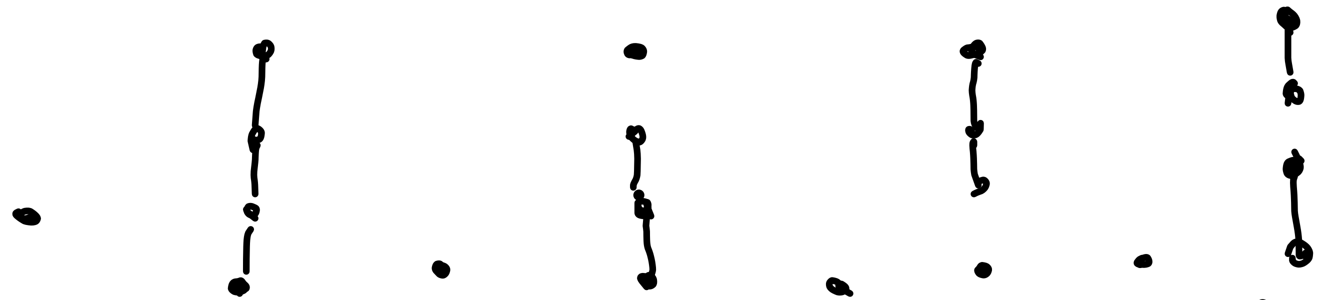
Tc. 28

$$\Rightarrow \quad f_n = F_{2n}$$

lze i jinak



f_4



4 $\quad$ $1\cdot 3$ $\quad$ $3\cdot 1$ $\quad$ $2\cdot 2$

$$f_4 = 4 + 1\cdot 3 + 3\cdot 1 + 2\cdot 2 + 2\cdot 1\cdot 1 + 1\cdot 2\cdot 1 + 1\cdot 1\cdot 2 + 1\cdot 1\cdot 1\cdot 1 = 21$$