3. demonstrační cvičení

Příklad 14. *Rozhodněte, zda je funkce*

$$f(x, y) = \sqrt{|xy|}$$

diferencovatelná v $[0, 0]$.

Ex. v okolí $[0,0]$ parc. derivace ?

$$f'_x(x_0, y_0) = \left(\sqrt{|\varepsilon x|}\right)_x = \sqrt{|\varepsilon|} \cdot \left(\sqrt{|x|}\right)' = \begin{cases} \sqrt{|\varepsilon|}\,\frac{1}{2\sqrt{|x|}} & x > 0 \\ \sqrt{|\varepsilon|}\,\frac{-1}{2\sqrt{|x|}} & x < 0 \end{cases}$$

$y_0 = \varepsilon$

pro $|x|$ velmi malou $\to$ $+\infty$ $x>0$, $-\infty$ $x<0$

$\Rightarrow$ parc. der. není spojitá v okolí počátku

Dok. že f není diferencovatelná

výpočtem směrové derivace pro $v=(1,1)$

a) z def.

$$d_v f(0,0) = \lim_{t\to 0} \frac{f(0+t\cdot v_1, 0+t\cdot v_2) - f(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{f(t,t)}{t} = \lim_{t\to 0} \frac{\sqrt{|t\cdot t|}}{t} = \lim_{t\to 0} \frac{|t|}{t} \text{ neex.}$$

neboť $\lim_{t\to 0^+} = 1$

$\lim_{t\to 0^-} = -1$

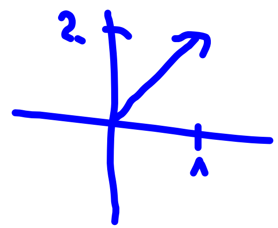
b) řez rovinou $y=x$

$f(x,x) = |x|$

↑

tato fce není diferencovatelná

v 0.



Příklad 15. *Pomocí diferenciálu přibližně vypočtěte:*

a) $\arcsin \frac{0,48}{1,05}$,

b) $1,04^{2,02}$.

a) $f(x,y) = \arcsin \frac{x}{y}$ v bodě $[0,5;1]$

$$f'_x(x,y) = \frac{\frac{1}{y}}{\sqrt{1-\frac{x^2}{y^2}}} = \frac{1/y}{\frac{1}{y}\sqrt{y^2-x^2}} = \frac{1}{\sqrt{y^2-x^2}}\Bigg|_{\substack{x=0,5\\y=1}}$$

$$f'_x(0,5;1) = \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{2}{\sqrt{3}}$$

$$f'_y(x,y) = \frac{-\frac{x}{y^2}}{\sqrt{1-\frac{x^2}{y^2}}} = \frac{-x/y^2}{1/y\cdot\sqrt{y^2-x^2}} = \frac{-\frac{x}{y}}{\sqrt{y^2-x^2}} = -\frac{x}{y\sqrt{y^2-x^2}}$$

$$f'_y(0,5;1) = \frac{-1/2}{\sqrt{3}/2} = -1/\sqrt{3}$$

$$f(0{,}48;\, 1{,}05) \approx f\left(\tfrac{1}{2};\, 1\right) + df\left(\tfrac{1}{2};\, 1\right)(-0{,}02;\, 0{,}05)$$

$$= f\left(\tfrac{1}{2};\, 1\right) + f_x\left(\tfrac{1}{2};\, 1\right)\cdot(-0{,}02) + f_y\left(\tfrac{1}{2};\, 1\right)\cdot(0{,}05) =$$

$$= \frac{\pi}{6} + \frac{2}{\sqrt{3}}(-0{,}02) - \frac{1}{\sqrt{3}}\cdot 0{,}05 =$$

$$= \underline{\underline{\frac{\pi}{6} - \frac{0{,}09}{\sqrt{3}}}}$$

b) $1{,}04^{2{,}02}$ – viz přednáška

Příklad 16. *Určete rovnici tečné nadroviny ke grafu funkce v daném bodě:*

a) $f(x,y) = x^2 + xy + 2y^2, \quad [x_0, y_0, z_0] = [1, 1, 4],$

b) $f(x,y) = \operatorname{arctg}\frac{y}{x}, \quad [x_0, y_0, z_0] = [1, -1, ?].$

a) tečná rovina má obecnou rovnici

$$f'_x \cdot (x - x_0) + f'_y (y - y_0) + f(x_0, y_0) = z$$

$$f'_x(1,1) = (2x+y)\big|_{1,1} = 3$$

$$f'_y(1,1) = (x+4y)\big|_{1,1} = 5$$

$$3(x-1) + 5(y-1) = 4$$

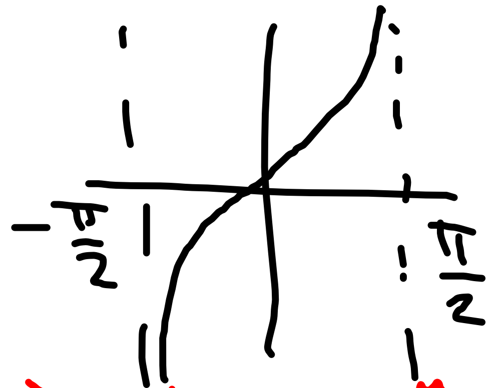
$$3(x-1) + 5(y-1) + 4 = z$$

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

tečná rovina v E_3.

b) $f(x,y) = \operatorname{arctg}\frac{y}{x}$ $\quad [x_0, y_0, z_0] = [1, -1, ?]$

$$z_0 = \operatorname{arctg}(-1) = -\frac{\pi}{4}$$



tečná rovina

$$z = -\frac{\pi}{4} + \frac{1}{2}\overbrace{(x-1)}^{dx} + \frac{1}{2}(y+1) = -\frac{\pi}{4} + \frac{x}{2} + \frac{y}{2}$$

$$f'_x(x,y) = \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} = \frac{-\frac{y}{x^2}}{\frac{x^2+y^2}{x^2}} = -\frac{y}{x^2+y^2}\Big|_{1,-1} = \frac{1}{2}$$

$$f'_y(x,y) = \frac{\frac{1}{\cancel{x}}}{1+\frac{y^2}{x^2}} = \frac{\frac{1}{\cancel{x}}}{\frac{x^2+y^2}{x^2}} = \frac{x^2}{\cancel{x}(x^2+y^2)}\Big|_{1,-1} = +\frac{1}{2}$$

Příklad 17. *Určete Taylorův polynom 2. stupně se středem v daném bodě:*

a) $\ln \sqrt{x^2 + y^2}$, $\quad [x_0, y_0] = [1, 1]$,

b) $x^{\frac{y}{z}}$, $\quad [x_0, y_0, z_0] = [1, 1, 1]$.

a) $f(1,1) = \ln\sqrt{2} = \ln 2^{1/2} = \frac{1}{2}\ln 2$

$f'_x(x,y) = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{x^2+y^2}$ $\qquad f'_x(1,1) = \frac{1}{2}$

$f'_y(x,y) = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{y}{x^2+y^2}$ $\qquad f'_y(1,1) = \frac{1}{2}$

$f''_{xy}(1,1) = -\frac{1}{2}$

$f''_{xx}(x,y) = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}\Big|_{1,1} = 0$

$f''_{xy}(x,y) = \frac{-2y\cdot x}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$ $\quad f''_{yy}(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}\Big|_{1,1} = 0$

$$f(x,y) = f(x_0,y_0) + f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0) +$$

$$+ \frac{1}{2}\Big(f''_{xx}(x_0,y_0)(x-x_0)^2 + 2 f''_{xy}(x_0,y_0)(x-x_0)(y-y_0)$$

$$+ f''_{yy}(x_0,y_0)(y-y_0)^2 \Big)$$

$$f(x,y) = \frac{1}{2}\ln 2 + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) -$$

$$- \frac{1}{2}\frac{1}{2} \cdot 2(x-1)(y-1)$$

$$(x-x_0, y-y_0)\begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}$$

b) $f(x,y,z) = x^{\frac{y}{z}} = e^{\frac{y}{z}\cdot \ln x}$

$f'_x = \frac{y}{z} \cdot x^{\frac{y}{z}-1} \Big|_{1,1,1} = 1$

$f'_y = \underbrace{e^{\frac{y}{z}\cdot \ln x}}_{x^{y/z}} \cdot \ln x \cdot \frac{1}{z} = x^{\frac{y}{z}} \cdot \ln x \cdot \frac{1}{z} \Big|_{1,1,1} = 0$

$f'_z = x^{\frac{y}{z}} \cdot \left(-\frac{y \ln x}{z^2}\right) \Big|_{1,1,1} = 0$

$f''_{xx} = \frac{y}{z} \cdot \left(\frac{y}{z}-1\right) \cdot x^{\frac{y}{z}-2} \Big|_{1,1,1} = 0$

$f''_{xy} = \frac{1}{z} \cdot x^{\frac{y}{z}-1} + \frac{y}{z} \cdot \left(x^{\frac{y}{z}-1}\right)'_y = \frac{1}{z} \cdot x^{\frac{y}{z}-1} +$

$+ \frac{y}{z} \cdot \left(x^{\frac{y}{z}-1}\right) \cdot \ln x \cdot \frac{1}{z} \Big|_{1,1,1} = 1$

$f''_{yy} = \ln x \cdot \frac{1}{z} \cdot x^{\frac{y}{z}} \cdot \ln x \cdot \frac{1}{z} \Big|_{1,1,1} = 0$

$$f''_{xz} = -\frac{y}{z^2}\cdot x^{\frac{y}{z}-1} + \frac{y}{z}\cdot x^{\frac{y}{z}}\left(-\frac{y\ln x}{z^2}\right)\Big|_{1,1,1} = 1$$

$$f''_{yz} = \ln x\left(\ldots\ldots \right)\Big|_{1,1,1} = 0$$

$$f''_{zz} = \ln x\left(\ldots\ldots \right)\Big|_{1,1,1} = 0$$

$$T_2 f_{1,1,1} = f(1,1,1) + df(1,1,1)\cdot(x-1,\ y-1,\ z-1) +$$

$$+\ \frac{1}{2}\cdot d^2f(1,1,1)(x-1, y-1, z-1) =$$

$$= 1 + (x-1) + \frac{1}{2}\cdot\left[2\cdot 1\cdot(x-1)(y-1) + 2\cdot(-1)\begin{matrix}(x-1)\\(z-1)\end{matrix}\right]$$

$$= x + \left((x-1)(y-1) - (x-1)(z-1)\right)$$