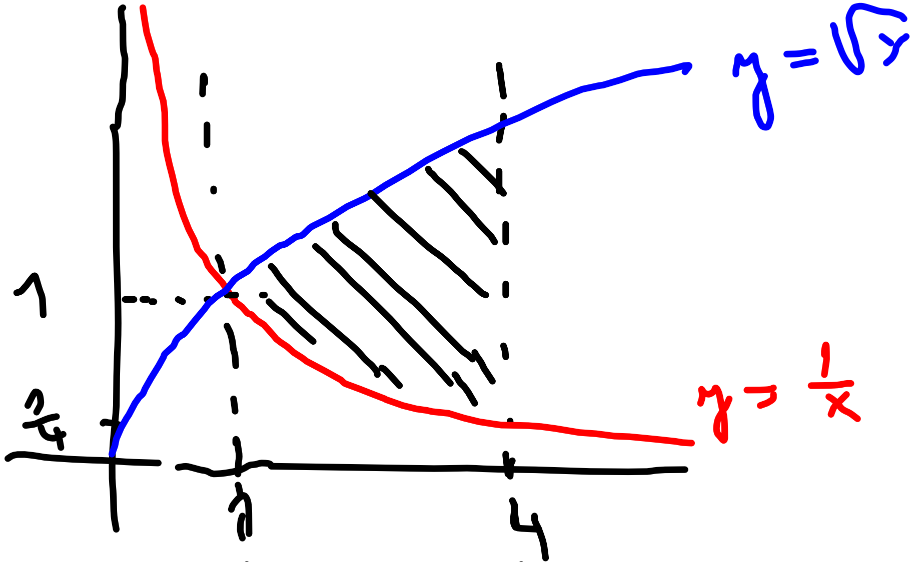
**Příklad 30.** *Vypočtěte* $\iint_M xy \,\mathrm{d}x\,\mathrm{d}y$, *kde* $M$ *je oblast* $1 \le x \le 4, \frac{1}{x} \le y \le \sqrt{x}$.




$$\int_1^4 \int_{1/x}^{\sqrt{x}} xy \, dy \, dx =$$

$$= \int_1^4 x \left( \int_{1/x}^{\sqrt{x}} y \, dy \right) dx = \frac{1}{2} \int_1^4 x \left[ y^2 \right]_{1/x}^{\sqrt{x}} dx =$$

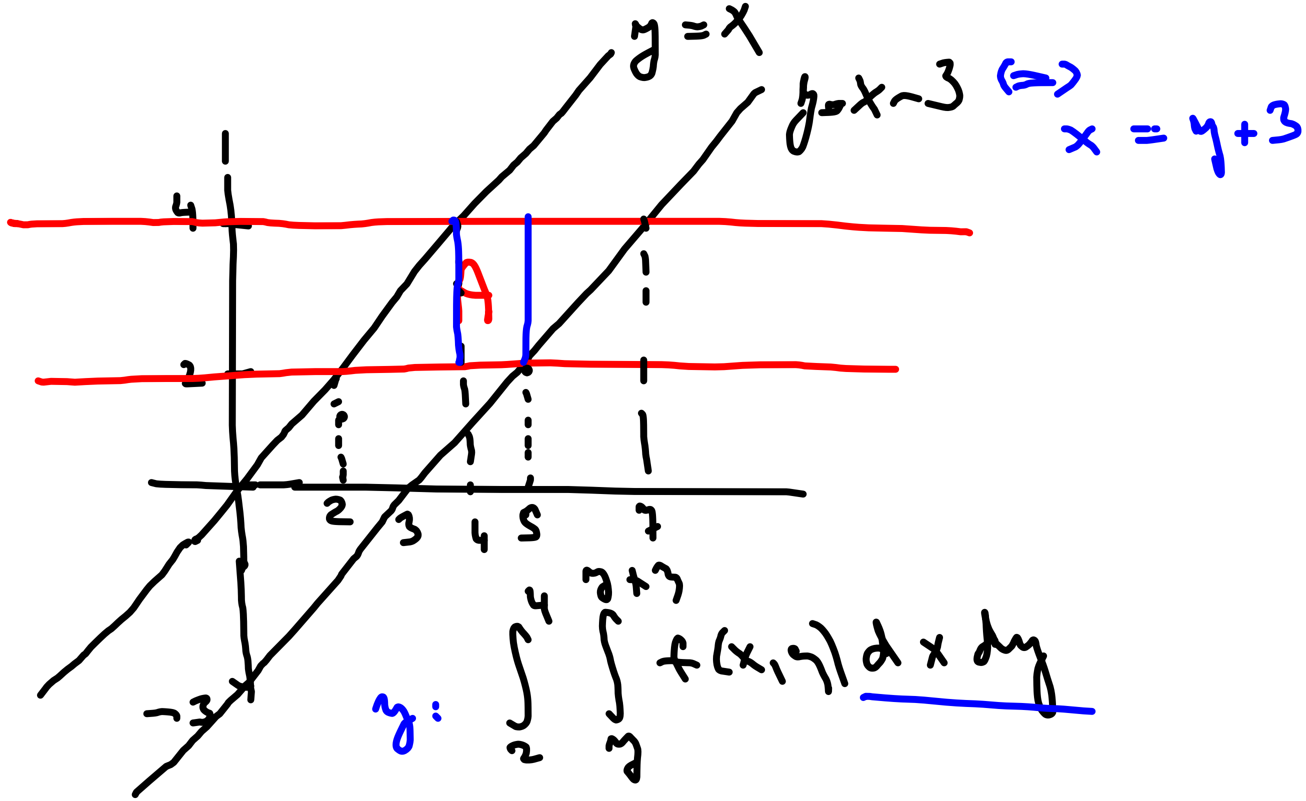
$$= \frac{1}{2} \int_1^4 x \left( x - \frac{1}{x^2} \right) dx = \frac{1}{2} \int_1^4 x^2 - \frac{1}{x} \, dx = \frac{1}{2} \left[ \frac{x^3}{3} - \ln|x| \right]_1^4 =$$

$$= \frac{1}{2} \left[ \frac{64}{3} - 2\ln 2 - \frac{1}{3} + 0 \right] =$$

$$= \frac{1}{2}\left(\frac{64}{3} - 2\ln 2 - \frac{1}{3} + 0\right) =$$

$$= \frac{21}{2} - \ln 2$$

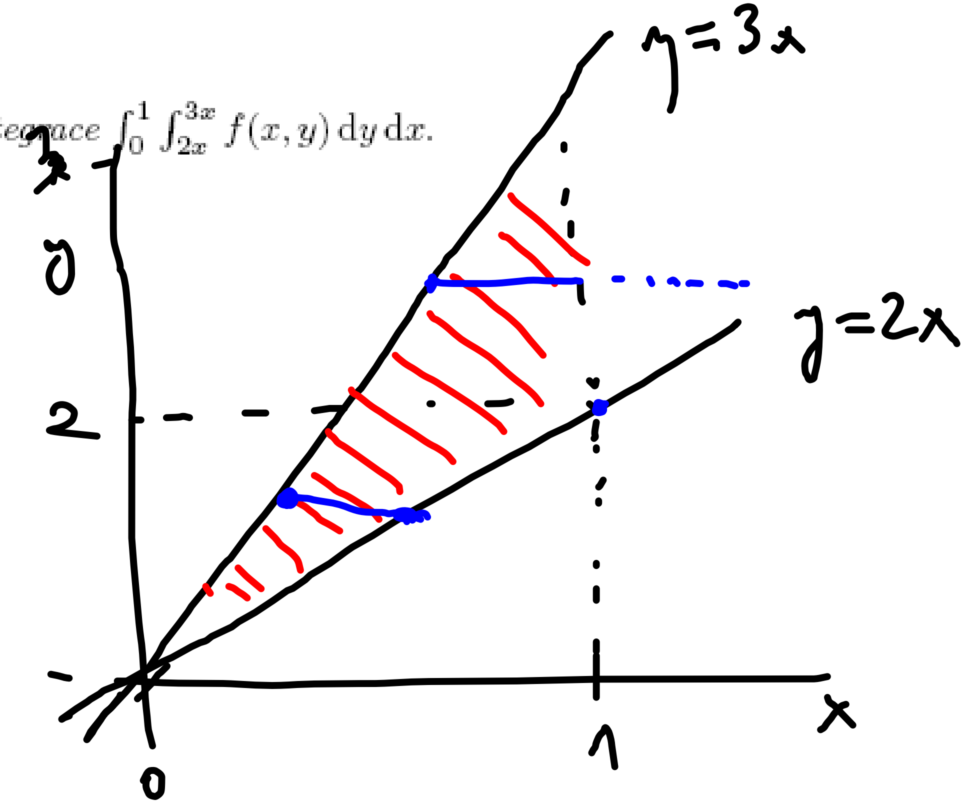
**Příklad 31.** *Převeďte dvojný integrál $\iint_A f(x,y)\,\mathrm{d}A$ na dvojnásobný (obě možnosti pořadí integrace) pro množinu $A$ ohraničenou přímkami $y = x, y = x-3, y = 2, y = 4$. Ověřte (přímo nebo s využitím SW např.* `MAW`*) rovnost výsledku pro konkrétní funkci $f(x,y) = y$.*




y: $\displaystyle\int_2^4 \int_y^{y+3} f(x,y)\,dx\,dy$

x: $\displaystyle\int_2^4 \int_2^x f(x,y)\,dy\,dx + \int_4^5 \int_2^4 f(x,y)\,dy\,dx + \int_5^7 \int_{x-3}^4 f(x,y)\,dy\,dx$ $\left[\frac{20}{3}\right]$

**Příklad 32.** *Zaměňte pořadí integrace* $\int_0^1 \int_{2x}^{3x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$.




$$0 \le y \le 3$$

$$y \le 2 \Rightarrow \frac{y}{3} \le x \le \frac{y}{2}$$

$$y \in (2,3\rangle \Rightarrow \frac{y}{3} \le x \le 1$$

$$\int_0^2 \int_{y/3}^{y/2} f(x,y)\,dx\,dy + \int_2^3 \int_{y/3}^{1} f(x,y)\,dx\,dy$$

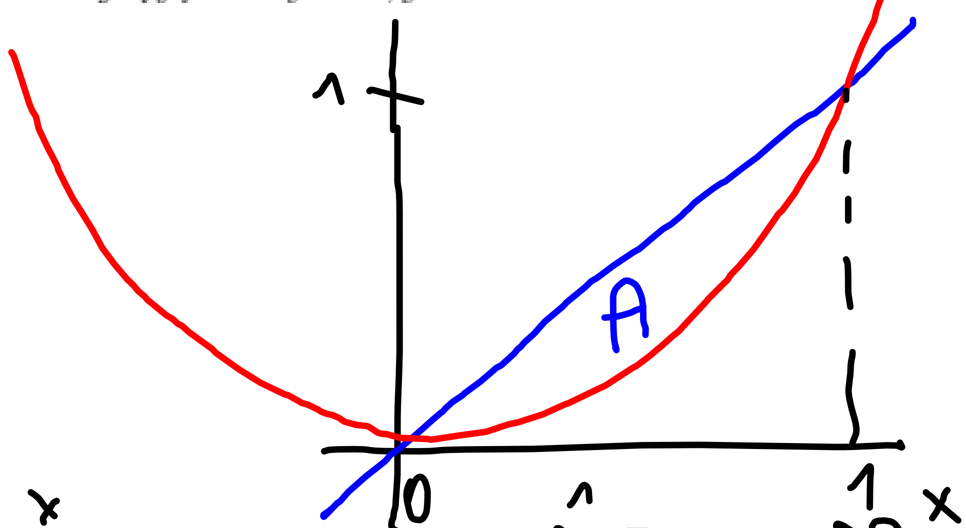
**Příklad 33.** *Vypočtěte integrál*

$$\iint_{.} (x + y)\,\mathrm{d}x\,\mathrm{d}y,$$

**Příklad 33.** *Vypočtěte integrál*

$$I=\iint_A (x+y)\,\mathrm{d}x\,\mathrm{d}y,$$

*kde A je ohraničena grafy funkcí* $y=x^2, y=x$.




$$I=\int_0^1\int_{x^2}^{x}(x+y)\,dy\,dx=\int_0^1\left[xy+\frac{y^2}{2}\right]_{x^2}^{x}dx=$$

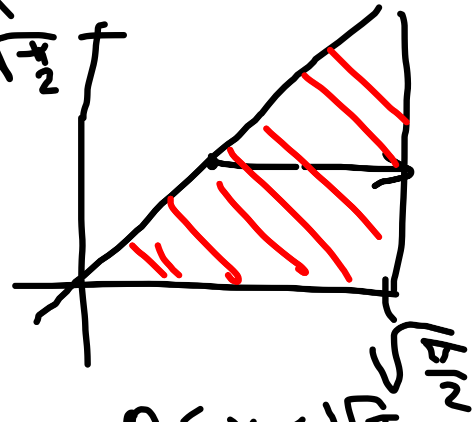
$$=\int_0^1\left(x^2+\frac{x^2}{2}-x^3-\frac{x^4}{2}\right)dx=\left[\frac{x^3}{3}+\frac{x^3}{6}-\frac{x^4}{4}-\frac{x^5}{10}\right]_0^1=$$

$$=\frac{1}{2}-\frac{1}{4}-\frac{1}{10}=\underline{\underline{\frac{3}{20}}}$$

**Příklad 34.** *Vypočtěte integrál*

$$I = \int_0^{\sqrt{\frac{\pi}{2}}} \int_y^{\sqrt{\frac{\pi}{2}}} y^2 \sin x^2 \, dx \, dy.$$

$$I = \int_0^{\sqrt{\frac{\pi}{2}}} y^2 \underbrace{\left( \int_y^{\sqrt{\frac{\pi}{2}}} \sin x^2 \, dx \right)}_{\text{neumíme!}} dx$$



Změníme pořadí integrace!

$0 \le x \le \sqrt{\frac{\pi}{2}}$

$0 \le y \le x$

$$\int_0^{\sqrt{\frac{\pi}{2}}} \int_0^{x} y^2 \sin x^2 \, dy \, dx = $$

$$= \int_0^{\sqrt{\frac{\pi}{2}}} \sin x^2 \int_0^{x} y^2 \, dy \, dx = \int_0^{\sqrt{\frac{\pi}{2}}} \sin x^2 \cdot x^3 \cdot \frac{2}{2 \cdot 3} \, dx =$$

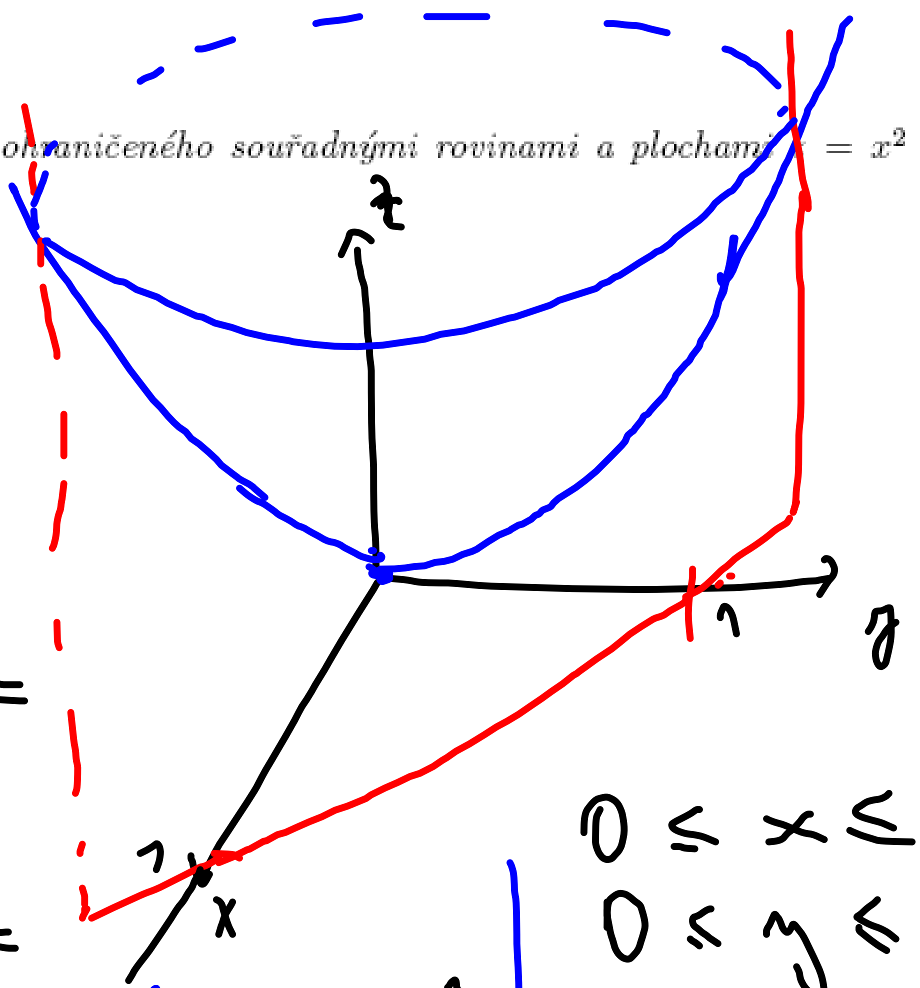
$$= \left| \begin{matrix} t = x^2 \\ dt = 2x \, dx \end{matrix} \right| = \int_0^{\pi/2} \sin t \cdot \frac{t}{6} \, dt =$$

$$\overset{\text{per partes}}{=} \left| \begin{matrix} u = t & u' = 1 \\ v' = \frac{\sin t}{6} & v = -\frac{\cos t}{6} \end{matrix} \right| =$$

$$= \frac{1}{6}\left( -\underbrace{\left[ t \cdot \cos t \right]_0^{\frac{\pi}{2}}}_{=0} + \int_0^{\pi/2} \cos t \, dt \right) =$$

$$= \frac{1}{6} \cdot \left[ \sin t \right]_0^{\frac{\pi}{2}} = \frac{1}{6}$$

**Příklad 35.** *Vypočtěte objem tělesa ohraničeného souřadnými rovinami a plochami* $z = x^2 + y^2$, $x + y = 1$.




$$\iiint_A dA$$

$$\int_0^1 \int_0^{1-x} \int_0^{x^2+y^2} dz\, dy\, dx =$$

$$= \int_0^1 \int_0^{1-x} x^2+y^2\, dy\, dx =$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx = \int_0^1 x^2(1-x) + \frac{(1-x)^3}{3}\, dx =$$

$0 \leq x \leq 1$

$0 \leq y \leq 1-x$

$0 \leq z \leq x^2+y^2$

$$= \frac{1}{3}\int_0^1 (3x^2 - 3x^3 + 1 - 3x + 3x^2 - \cancel{x^3})\, dx = \frac{1}{3}\int_0^1 -4x^3 + 6x^2 - 3x + 1\, dx =$$

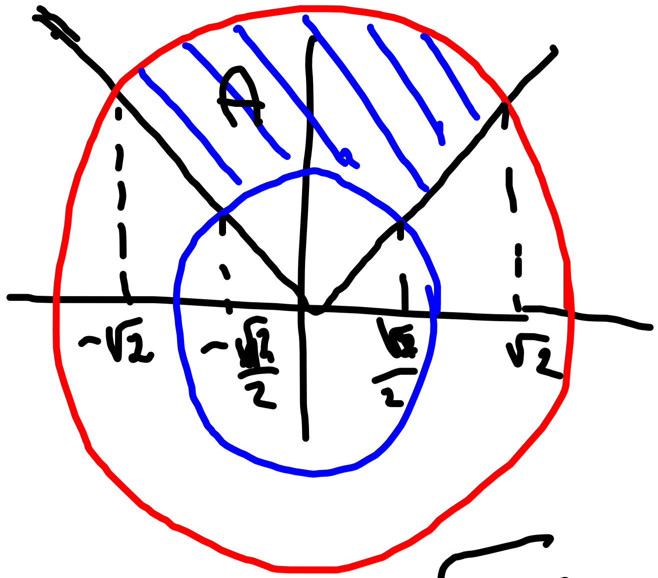
$$= \frac{1}{3}\left[-x^4 + 2x^3 - \frac{3x^2}{2} + x\right]_0^1 =$$

$$= \frac{1}{3}\left(-1 + 2 - \frac{3}{2} + 1\right) = \underline{\underline{\frac{1}{6}}}$$

**Příklad 36.** *Vypočtěte integrál*

$$\iint_A 2(x^2+y^2)\,\mathrm{d}A,$$

*kde* $A = \{[x,y] \in \mathbb{R}^2 : 1 \le x^2+y^2 \le 4, y \ge |x|\}$.




$$\int_{-\sqrt{2}}^{-\frac{\sqrt{2}}{2}} \int_{-x}^{\sqrt{4-x^2}} 2(x^2+y^2)\,dy\,dx + \dots$$

$y \le \sqrt{4-x^2}$

dohl nač?

To se nám nechce počítat!

Polární souřadnice!

$y = -x$

$x^2+y^2 = 1$

$2x^2 = 1$

$$x = \pm\frac{\sqrt{2}}{2}$$

$x = r\cos\varphi$ $\quad D^1 f = r$

$y = r\sin\varphi$

Transformace : $r \geq 0$

$$\left[ \begin{array}{l} r = \sqrt{x^2+y^2} \\ \operatorname{tg}\varphi = \frac{y}{x} \end{array} \right] \quad 1 \leq x^2+y^2 \leq 4 \Rightarrow 1 \leq r^2 \leq 4 \Leftrightarrow 1 \leq r \leq 2$$

$$y \geq |x| \Rightarrow \frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}$$

$$\int_1^2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2r^2 \cdot r \, d\varphi \, dr = \int_1^2 \int_{\frac{\pi}{4}}^{3\pi/4} 2r^3 \, d\varphi \, dr =$$

$$= 2\int_1^2 r^3 \, dr \cdot \int_{\pi/4}^{3\pi/4} d\varphi = 2 \cdot \frac{\pi}{2} \cdot \left[\frac{r^4}{4}\right]_1^2 =$$

$$= \underline{\pi \cdot \left(4 - \frac{1}{4}\right)}$$

**Příklad 37.** *Spočtěte integrál*

$$\int_0^1 \int_{-\sqrt{x-x^2}}^{\sqrt{1-x^2}} \mathrm{d}y\,\mathrm{d}x.$$

polární souřadnice

$I_1: 0 \le \varphi \le \frac{\pi}{2}, \quad 0 \le r \le 1$

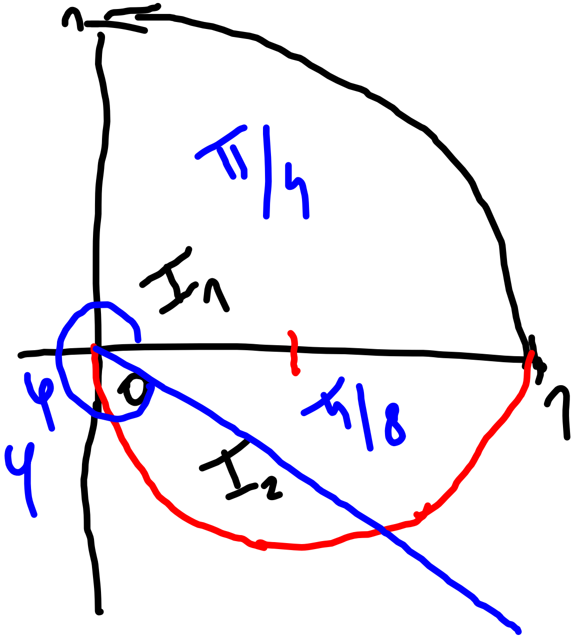
$I_2: \frac{3\pi}{2} \le \varphi \le 2\pi, \quad 0 \le r \le \cos\varphi$

$y^2 \le x - x^2$

$r^2 \sin^2\varphi \le r\cos\varphi - r^2\cos^2\varphi$

$r^2 \le r\cos\varphi$

$\underline{r \le \cos\varphi}$




$y^2 \le 1 - x^2$

$x^2 + y^2 \le 1$

$0 \ge y \ge -\sqrt{x - x^2}$

$\boxed{y^2 \le x - x^2}$

$x^2 + y^2 \le x$

$(x - \frac{1}{2})^2 + y^2 \le \frac{1}{4}$

$$\int_0^{\frac{\pi}{2}}\int_0^1 r\,dr\,d\varphi + \int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\cos\varphi} r\,dr\,d\varphi =$$

$$= \int_0^{\pi/2} \frac{1}{2}\,d\varphi + \int_{\frac{3\pi}{2}}^{2\pi} \frac{\cos^2\varphi}{2}\,d\varphi =$$

$\cos^2\varphi - \sin^2\varphi = \cos 2\varphi$

$2\cos^2\varphi - 1 = \cos 2\varphi$

$$= \frac{\pi}{4} + \frac{1}{4}\int_{\frac{3\pi}{2}}^{2\pi} \cos 2\varphi + 1\,d\varphi =$$

$$= \frac{\pi}{4} + \frac{1}{4}\left[\frac{1}{2}\sin 2\varphi + \varphi\right]_{\frac{3\pi}{2}}^{2\pi} = \frac{\pi}{4} + \frac{1}{4}\left(2\pi - \frac{3\pi}{2}\right) =$$

$$= \frac{\pi}{4} + \frac{\pi}{8} = \underline{\underline{\frac{3\pi}{8}}}$$

Př 37 lze řešit i transformací

$$x = r\cos\varphi + r$$
$$y = r\sin\varphi$$

Pak $DF = \begin{vmatrix} \cos\varphi + 1 & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r(1+\cos\varphi)$

.... 2. podmínka dá $\frac{1}{2} \geq r$

$$\int_{\pi}^{2\pi} \int_{0}^{\frac{1}{2}} r(1+\cos\varphi)\, dr\, d\varphi = \dots = \frac{\pi}{8}$$

(obsah „dolního" půlkruhu).

**Příklad 38.** *Spočtěte integrál* $\iint_A \sqrt{xy}\,\mathrm{d}x\,\mathrm{d}y$ *,kde množina A je ohraničena křivkami*

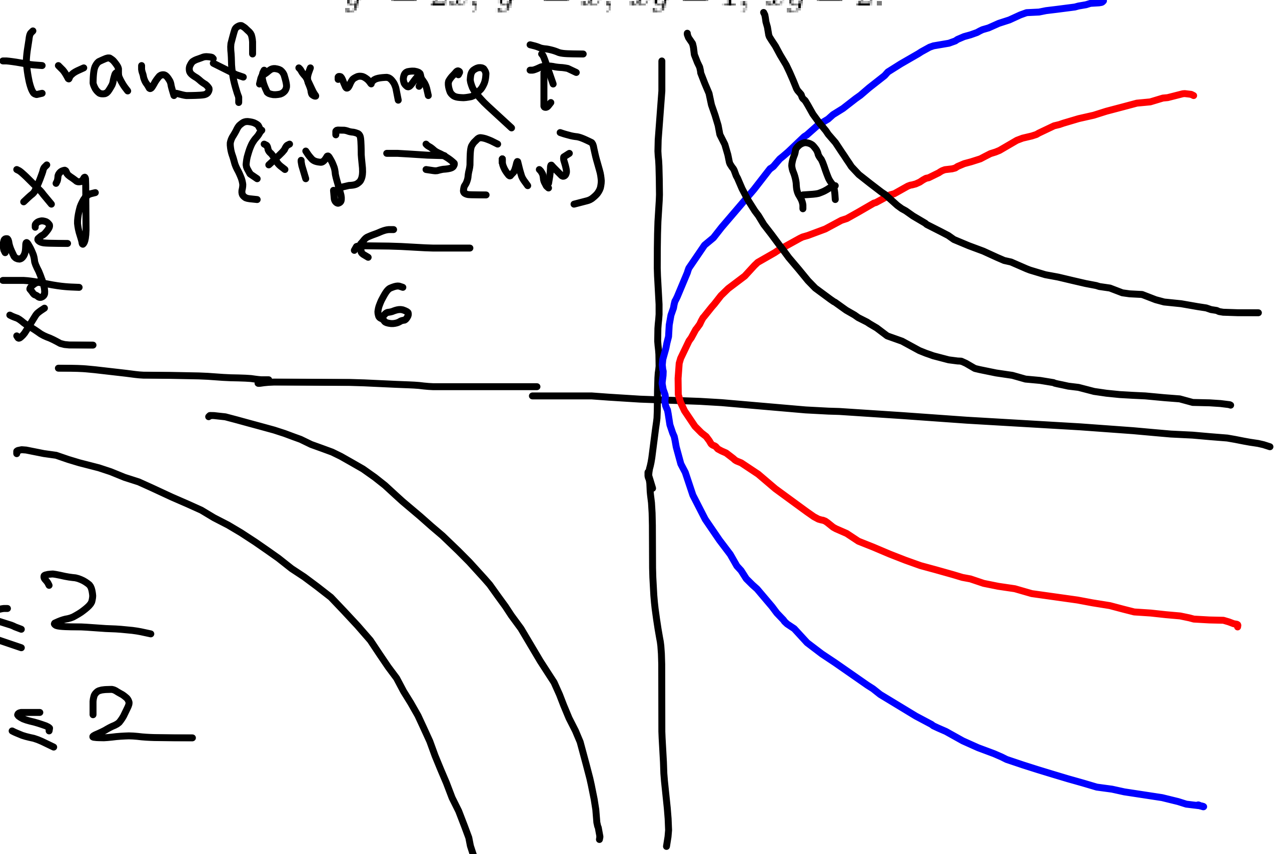
$$y^2 = 2x,\ y^2 = x,\ xy = 1,\ xy = 2.$$

spec. transformace F

$[x,y] \xrightarrow{F} [u,v]$, $\xleftarrow{G}$

$u = xy$

$v = \dfrac{y^2}{x}$




$1 \le u \le 2$

$1 \le v \le 2$

$$DF = \begin{vmatrix} y & x \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = \frac{2y^2}{x} + \frac{y^2}{x} = \frac{3y^2}{x} \Rightarrow DG = \frac{x}{3y^2} = \frac{1}{3v}$$

$$\iint_A \sqrt{xy}\,dx\,dy =$$

$$= \int_1^2 \int_1^2 \sqrt{u} \cdot \frac{1}{3v}\,du\,dv = \text{snadno} =$$

$$= \frac{2}{9}\left(2\sqrt{2}-1\right)\ln 2 .$$